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NOTE-BOOK

ON

PRACTICAL SOLID

OR

DESCRIPTIVE GEOMETRY

NOTE-BOOK

ON

PRACTICAL SOLID

OR

DESCRIPTIVE GEOMETRY

CONTAINING

PROBLEMS WITH HELP FOR SOLUTIONS.

BY

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PREFACE.

In teaching a large class, if the method of lecturing and demonstrating from the black-board only is pursued, the more intelligent students have generally to be kept back, from the necessity of frequent repetition, for the sake of the less promising; if the plan of setting problems to each pupil is adopted, the teacher finds a difficulty in giving to each sufficient attention. A judicious combination of both methods is doubtless the best, though this is not always easy of attainment in working with numbers—the use of this book may help in accomplishing it.

It is suggested that at the beginning of a chapter, and in some cases with each problem, the teacher should give a black-board explanation, carefully pointing out any fresh steps, before sending his pupils to their work. The number of examples in each chapter to be worked out by the student is, of course, left to the teacher's judgment of the progress and requirements of his pupil.

The student will here be assisted, it is believed, because he will find help given when necessary, and thus will often be able to go on with his work by himself, with the satisfaction of also feeling that he is progressing systematically. Moreover he will become practised in dealing with written questions—a point not to be lost sight of by those who are preparing for Examinations.

The *Second Edition* has been enlarged by the addition of chapters on the straight line and plane with explanatory diagrams and exercises, on tangent-planes, and on the cases of the spherical triangle.

E. G.

It is hoped that the work, thus rendered more complete, may prove still more useful as a class-book and means of self-instruction to the various and constantly increasing classes of students for whom it is designed. It was originally intended as an aid in teaching the Mechanical Drawing Class at the Royal School of Mines from Professor Bradley's *Elements of Practical Geometry.* The authors of this work were associated with him in his duties at King's College, London, and the Royal Military Academy, and learnt practically the value of his treatise; but the cost of that work has rendered it inaccessible to many for whom the present book may be available.

A greater number of diagrams have not been added, in order that students may be thrown upon their own resources, and encouraged to consider the *principles* upon which their work proceeds, more than they would probably do if there were figures always at hand for reference.

October, 1871.

ADVERTISEMENT TO THE FOURTH EDITION.

A STEADY demand for the "NOTE-BOOK" having shown that the principles on which it was designed met with the recognition of teachers and students, the successive editions have been enlarged to satisfy the requirements of those competing in the various public examinations, as well as to adapt it to the gradual development of the class at the Royal School of Mines.

In preparing the *Fourth Edition* the original scope of the book has been considerably extended. The aim being to make it, without prejudice to the elementary character of its earlier portions, an exhaustive *vade-mecum* for the working student. Attention may, therefore, be called to the following points in the treatment of the present edition which have consequently become more prominent:—

1. While retaining its original form of a classified collection of "Problems with Help for Solutions," the book has been placed upon a more independent and scientific footing; by adding the necessary definitions and theorems, and introducing a large amount of matter of a general character, not elsewhere accessible, in a cheap and compendious form, to the majority of English students.

2. A large number of carefully selected problems have been added to illustrate the leading principles and afford specific exercises for self-examination, while in writing the solutions the constructions have been studiously varied so as to bring together as many as possible of the methods likely to be of practical utility to the scientific draughtsman.

3. Diagrams have been purposely eschewed and a systematic attempt made to reduce the whole subject, as far as possible, to a purely verbal form—sufficient description of the various constructions being given to enable a student who works steadily through the earlier chapters to build up, under his own hand and eye, illustrative drawings for himself. The verbal formulæ, besides being much less fatiguing to peruse than the plates usually accompanying books on this subject, soon become immeasurably more powerful and suggestive helps, and leave a wider field for the exercise of the learner's ingenuity. Moreover, the style of treatment adopted has rendered it possible to compress into a very small compass a quantity of matter which would otherwise have augmented the bulk and cost of the book so as to have placed it out of the reach of many whose wants it is intended to supply.

4. Some of the more advanced problems have been dealt with in general terms and very briefly, experience having shown that when a student's own thought or theoretical studies have prepared him for the necessary geometrical conceptions, these higher developments are really the most easy, and a hint is almost all that is required as a starting-point for fresh knowledge—which is not so much learned as self-evolved when once the mind is set upon the proper track.

June, 1880.

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ELEMENTARY

SOLID OR DESCRIPTIVE GEOMETRY.

INTRODUCTORY EXPLANATIONS.

"The object of Descriptive Geometry is the invention of methods by which we may represent upon a plane having only two dimensions, namely length and breadth, the form and position of a body which possesses three dimensions, namely length, breadth, and height."

HALL'S *Elements of Descriptive Geometry.*

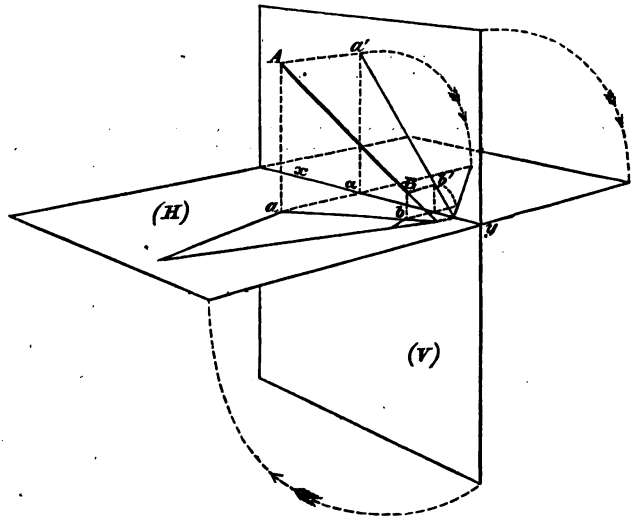
FOR this purpose two planes, called the co-ordinate planes, are conceived at right angles to one another, intersecting in a line called the *ground line, xy*, and named from their usual positions the *horizontal* and *vertical planes of projection.* See Fig. 1.

Drawings or *projections* on the horizontal plane are called *plans;* on the vertical plane *elevations.*

The *plan* or *horizontal projection* of any point *A* in space, is the foot of the perpendicular let fall from point *A* to the

horizontal plane, and is marked a; the *elevation* of A is marked a', and is the point of intersection of the perpendicular from A to the vertical plane. See Fig. 1.

Fig. 1.



The *projection* of a line may be defined to be the sum of the projections of its points.

Forms possessing three dimensions, length, breadth, and height, require to be represented or projected upon both planes of projection, so that "height," as well as "length" and "breadth," may be exhibited or determined by the drawings.

But as we can have for drawing only one plane, that is, one flat sheet of paper, not two at right angles to each other, the vertical plane is supposed to rotate backwards upon xy,

its intersection with the horizontal plane, as upon a hinge, until it coincides and forms one plane with the horizontal. The plan a and elevation a' of any point A *will then lie in the same perpendicular to the ground line.*

This perpendicular, called the "projecting line" of the point A, will thus form a *locus* of points a and a', that is to say, must contain points a and a'.

The distance of a from xy shows the distance of the point A from the vertical plane.

The distance of a' from xy shows the distance of the point A from the horizontal plane, i.e. its height.

The object of all science is exact knowledge, and as this is impossible of attainment unless language be made to keep pace in precision with the advance of thought, it has been considered desirable to rigidly define at the outset all the important terms that have a meaning peculiar to the subject. It is not intended however that the student at the very commencement of his labours shall learn these by rote, but he is earnestly requested to turn back from time to time, and assure himself that he has at least mastered the *significance* of such technical terms as form part of the description of the problem he is working. Many a difficulty will be got over by a timely reference to the definitions, while full mastery of the language of the subject is the only way to a thorough grasp of its principles.

DEFINITIONS AND THEOREMS.

The object of Solid Geometry is systematised knowledge of the geometrical properties of forms having three dimensions.

Descriptive Geometry deals directly, with the relations of these forms to two co-ordinate planes at right angles, and indirectly, with their relations to one another. The object of the science being to systematise our knowledge of the *methods* by which, and the *principles* upon which, these forms of three dimensions can be represented upon a plane or surface of two dimensions, so that, from the representation alone, the real shape, size, and relationship of the forms may be unerringly inferred.

The *method* of Descriptive Geometry is almost wholly a method of projections, as laid down in the definitions that follow.

The *principles* are involved in the application of the method to the solution of problems, many of the leading results of which find expression in the subjoined theorems.

PRELIMINARY DEFINITIONS.

1. The *projection* of a *point* on a plane is the foot of the perpendicular let fall from the point to the plane.

2. The *projection* of a *line* (straight, curved, or broken), *figure*, or *form* is the sum of the projections of its points.

3. The *plan* of a point, line, figure, or form is its projection on the horizontal plane; its *elevation* is its projection on the vertical plane.

4. The perpendicular which gives the projection of a point on a plane, is called the *projector* of the point, and the plane on which the point is projected, is called the *plane of projection* of the point. The *vertical projector* is the one drawn to the vertical plane; the *horizontal projector* the one drawn to the horizontal plane of projection.

5. The surface containing all the projectors of a straight or curved line, is called the *projecting surface* of the line. The intersection of this surface with the plane of projection coincides with the projection of the line upon that plane.

6. A *trace* is the intersection of a straight line or of a surface (plane or curved) with, unless otherwise stated, one of the planes of projection. The *horizontal trace* (H.T.) is the intersection with the horizontal plane (H.P.); the *vertical trace* (V.T.) is the intersection with the vertical plane (V.P.).

Straight lines are represented either by their traces or by the method of projections. By the latter when *finite*, and by the former when the length is *indefinite* or disregarded for the purposes of the problem. Similarly, *figures* are represented by their projections, and *planes*, indefinitely extended, by their traces. *Curved* surfaces are frequently represented by a combination of both. The traces of an indefinitely extended straight line, or of a plane, are always sufficient to determine it. The traces of a curved surface are not alone sufficient to determine it.

Note. The projections made in Descriptive Geometry, are called "*orthographic*" projections, to distinguish them from *radial* or *perspective* projections.

7. A *section plane* is a plane cutting through a form. The "*section*" is the *trace* of the form on this plane.

Planes may be *vertical*, *horizontal*, *inclined*, or *oblique*.

8. A *vertical plane* is one at right angles to the horizontal plane. It may make with the vertical plane of projection any angle between 0° and 90°. In the former position

it will be called a *parallel vertical* plane, and in the latter, a *right vertical* plane. See Theorems II. and III., and Fig. 5, planes $n\beta o'$ and $h\delta v'$.

9. A *horizontal plane* is one at right angles to the vertical plane and parallel to the horizontal plane of projection.

10. An *inclined plane* is one at right angles to the vertical plane and *not* parallel to the horizontal plane. Ex. qlm'. Fig. 5.

11. An *oblique plane* is one not at right angles to either plane of projection, *i.e.* is a plane inclined to both. See qlm', Fig. 2, and hav', Fig. 5. A plane inclined to both coordinate planes and parallel to xy, is called a *parallel oblique* plane. See plane rs, $t'u'$, Fig. 5. An oblique plane equally inclined to both planes of projection is called a *right oblique*.

12. A plane is said to be "*constructed*" or "*rabatted*," when it is rotated, with all its contained points, lines, and figures upon it, about one of its traces into the corresponding plane of projection.

13. A *development* is the figure produced by unrolling or laying out in one plane the surface or surfaces of a form. All surfaces are not *developable*, *i.e.* cannot be laid flat in one plane without tearing or doubling. *All* forms bounded by *plane* surfaces, and *some* bounded by *curved* surfaces, are developable. See Chap. VII., *Curved Surfaces*.

THEOREMS.

THEOREM I. The *traces* of a *straight line* are *points*. A line parallel to either plane of projection has no trace in the plane to which it is parallel, and therefore, a line parallel to

xy (the line in which the co-ordinate planes of projection intersect), being parallel to both planes of projection, will have no trace at all.

THEOREM II. The traces of a *plane* are *straight lines*, and, unless the plane is parallel to xy, the traces will meet at some point in that line. A plane parallel to xy, and not perpendicular to either plane of projection, will have for its traces two straight lines parallel to xy. A plane parallel to xy, and at right angles to one of the planes of projection, will be consequently parallel to the other plane of projection and will have no trace in that plane. Figs. 2 and 5.

THEOREM III. If the horizontal trace of a plane is at right angles to xy, the plane is perpendicular to the vertical plane of projection. If the vertical trace of the plane is at right angles to xy, the plane is perpendicular to the horizontal plane of projection. Fig. 5 planes qlm' and $n\beta o'$.

THEOREM IV. The two projections of a point are in the same *projecting line* or perpendicular to xy, after the vertical plane has been turned into the horizontal. Fig. 3.

THEOREM V. The perpendicular from the plan of a point to xy is parallel to the vertical projector of the point, and is equal to the distance of the point from the vertical plane. Similarly, the perpendicular from the elevation of a point to xy is parallel to the horizontal projector of the point, and is equal to the distance of the point from the horizontal plane.

THEOREM VI. When two planes are at right angles, every line drawn from a point in one of them perpendicular to the other, lies wholly in the former, and meets the latter

in, and is perpendicular *to*, the line of intersection of the planes. Whence it follows :—

(1) A point in one plane of projection has its projection on the other in *xy*. That is, if the point is *in* the vertical plane its *plan* is in *xy*. If in the horizontal plane its *elevation* is in that line.

(2) If a given plane be perpendicular to one plane of projection, it contains all the projectors drawn from any points, lines, or figures, in that plane, to the plane of projection to which the given plane is perpendicular. That is to say, the given plane is the projecting plane of all figures lying in it. The trace of this plane on the plane of projection to which it is at right angles, contains the projections on this plane of all points, lines, or figures lying in the given plane.

(α) Thus, if an inclined plane be perpendicular to the vertical plane of projection, the elevations of all points in the inclined plane will be in its *vertical trace*. *E.g.* A circle of 3 inches diameter lying in a plane inclined at 30° to the horizontal plane and perpendicular to the vertical plane, has a segment of the vertical trace, 3 inches long, for its *elevation*.

(β) A plane inclined to the vertical plane of projection and perpendicular to the horizontal plane, has the plans of all points, lines, and figures lying in it projected in its *horizontal trace*. Thus, the plans of the vertical faces of a right prism resting on its base coincide with the lines forming the sides of that base.

THEOREM VII. The projection of a straight line on a plane is a straight line. The projections upon the same plane of parallel straight lines are parallels.

The intersection of a plane with a series of parallel planes will be a series of parallel lines, and the projections of these intersections will, according to this theorem, be parallels.

THEOREM VIII. The orthographic projection of a finite straight line is equal to the real line if the latter is parallel to the plane of projection. The magnitude of the projection diminishes as the angle of inclination of the line increases, and becomes a minimum, *i.e.* a point, when the angle of inclination reaches 90°, or when the line is perpendicular to the plane.

The projection of a rectilineal angle on a plane parallel to that of the angle, is an angle equal to the given one.

The projection, on the same plane, of the circular arc subtending that angle will also be equal to that of the arc. See Figs. 21 and 22.

Generally, the projection of any plane figure on a plane parallel to it is a figure equal in all respects to the figure of which it is the projection. And the area of the projection of a given figure decreases as the angle between the plane of projection and the plane of the figure increases,—becoming a minimum, *i.e.* a straight line, when this angle is a right angle.

NOTE. This is a principle often taken advantage of in finding the true shape of sections. See Prob. 1, Chap. I.

THEOREM IX. The projections of a perpendicular to a plane are perpendiculars to the traces of the plane. The *plan* is at right angles to the *horizontal* trace; the *elevation* is at right angles to the *vertical* trace.

THEOREM X. If a straight line lies in a plane and has traces (Theorem I.), they will be in those of the plane. Thus a plane can be drawn to contain a given line by taking the traces of the plane through those of the line, when the latter are determinable.

THEOREM XI. The angle which the tangent plane to a right cone makes with the plane of the base of the cone, is measured by the plane angle which a generatrix of the cone makes with the base, or is the complement of the angle between a generatrix and the axis.

THEOREM XII. If a plane touches a conic surface and a second plane intersects them, the *trace* of the tangent plane on the intersecting one is a tangent to the trace of the conic surface on the same plane. Generally, any plane cutting a curved surface and its tangent plane through a point of contact of the tangent plane with the surface, will intersect the tangent plane in a line which is a tangent to the curve in which the cutting plane intersects the surface.

THEOREM XIII. The tangent plane at any point of a curved surface contains the tangent lines drawn at the same point to all the lines traced on the surface at that point.

THEOREM XIV. The tangent plane to a cone or a cylinder at a given point is the tangent plane to the surface at every other point in the generatrix passing through the given point, and all the normals to the surface along the generatrix are in a plane containing the generatrix and perpendicular to the tangent plane.

THEOREM XV. Parallel sections of a conic surface are similar but unequal curves.

THEOREM XVI. Points in the generatrix of a surface of revolution describe circles in planes perpendicular to the axis of revolution whose centres are the points of intersection of the axis with their respective planes.

THEOREM XVII. All sections of a surface of revolution made by planes containing the axis are similar and equal curves.

THEOREM XVIII. A plane which bisects a chord of a sphere at right angles passes through the centre of the sphere.

THEOREM XIX. The section of a sphere by a plane passing through its centre, is a circle whose radius is equal to the radius of the sphere—hence called a *great circle.* Any other plane section cuts a *small circle.*

THEOREM XX. When a plane is "*constructed*" or "*rabatted*" (Def. 12) about one of its traces every point in it will have for its *locus* a straight line drawn at right angles to the trace used through the projection of that point on the plane of projection containing the trace.

Thus if the *horizontal* trace be used as the axis of "rabatment" or "construction," the *loci* of all points in the plane will be the perpendiculars to the trace through their respective *plans.*

The reason for this relation or "locus" will be obvious by considering that every point in a "constructed" plane must describe a circle about the trace forming the axis, and that the projecting plane of the path of the point is a plane perpendicular to that trace.

These *loci* are made use of, for example, in determining the true shape of a section, and the projection of a frustum

of a solid on a plane which passes through it. The solution of many problems whose data lie in one plane is often greatly facilitated by this method of rabatments.

DEFINITIONS ILLUSTRATED.

The co-ordinate planes, upon which the projections are made, intersect at right angles in a line called the *ground line* (*xy*), and are named from their usual positions the *horizontal* and *vertical planes of projection.* Fig. 1.

The four dihedral angles formed by the intersecting planes of projection are known as the 1st, 2nd, 3rd, and 4th.

The 1st dihedral angle is that contained between the upper face of the H.P. and the front face of V.P.

The 2nd is that between the upper face of the H.P. and the back face of V.P.

The 3rd is between the lower face of H.P. and the back face of V.P.

The 4th between the lower face of H.P. and the front face of V.P.

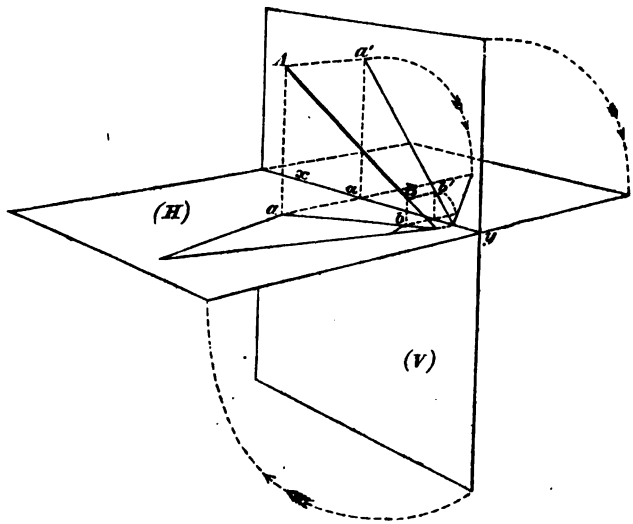
Notation.

A *point in space* is indicated by a capital letter *A*, its *projections* by italics *a* and *a'*: the accented letter denoting the vertical projection or elevation of the point, and the unaccented italic the plan. Figs. 1 and 3.

Similarly, a *line in space* is denoted by capital letters, as; *AB*: and its projections by italics, as; *ab*, *a'b'*. Figs. 1 and 4.

A plane is denoted by three letters, one on each trace and the third on the point of intersection in *xy*. Figs. 2 and 5.

Fig. 1.



Note. Figs. 1 and 2 are merely pseudoperspective sketches illustrative of the elementary principles and definitions, and must on no account be mistaken for the kind of drawing to be made in the solution of the problems. Figs. 3, 4, and 5 are orthographic illustrations of the mode of determining points, lines, and planes, and may be taken as elementary types of the kind of drawing to be made.

FIG. 1. The points aa' are the *projections* of the point A. Def. 1.

The lines ab, $a'b'$ are the *projections* of the line AB. Def. 2. The line ab is the *plan*, and $a'b'$ the *elevation*. Def. 3.

The perpendiculars Aa, Aa', and Bb, Bb', are the *projectors* of the points A and B; Aa, Bb, and Aa', Bb', being the *horizontal* and *vertical projectors* of the points A and B respectively, and the planes H. and V. the *horizontal* and *vertical planes of projection*. Def. 4.

The planes AB, ab, AB, $a'b'$, containing the given straight line AB and the projectors Aa, Aa', and Bb, Bb', are the *projecting planes* of the line AB, and the intersection of these planes with the planes of projection give the lines ab, $a'b'$ which are the projections of the lines AB. Def. 5.

As our drawings can only be made conveniently on one plane, we assume that the vertical plane, with the *vertical projection* on it, is rotated backwards in the direction shown by the arrows, about xy as an axis, until it coincides with the horizontal plane. In this position the *vertical* is oftentimes more or less superposed upon the *horizontal projection*, and as in all our reasonings upon this subject it is necessary to be clear as to what points, lines, &c. are in the vertical, and what in the horizontal planes, some conventional system of *notation*, such as that described above, has to be adopted. When the planes are thus rotated the *plan* a and the *elevation* a' of any point A *will lie in the same perpendicular to the ground line*, xy. Theorem IV.

This perpendicular is therefore a *locus* of points a and a', that is, will contain a and a'.

The distance of a from xy shows the distance of the point A from the vertical plane; and the distance of a' from xy shows the distance of the point A from the horizontal plane, *i.e.* its height. Thus aa is parallel and equal to Aa', and $a'a$ is parallel and equal to Aa. Theorem V.

Fig. 2.

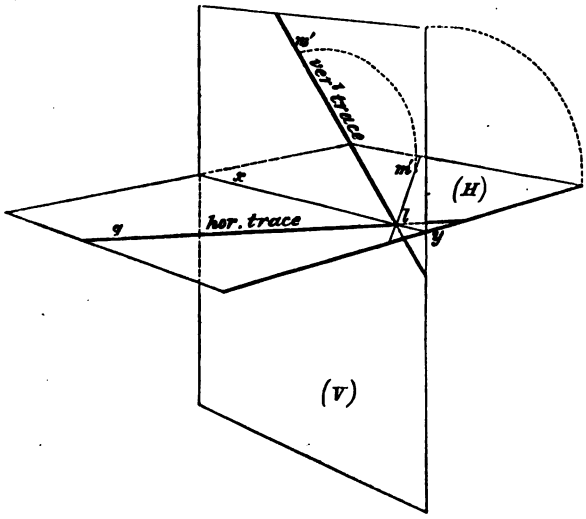
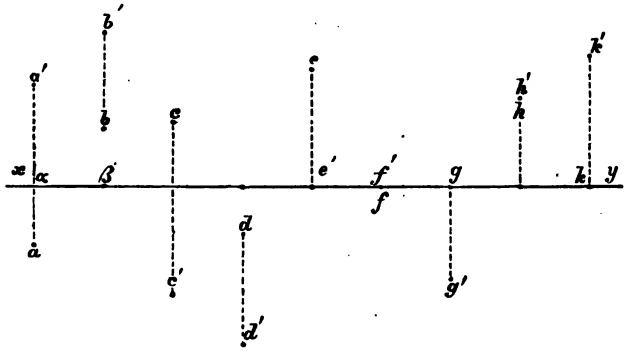


Fig. 2 shows the *traces* (Def. 6) of an "oblique" plane (Def. 11) meeting in xy (Theorem II.). The traces of every plane which is not a *parallel vertical*, a *horizontal* or a *parallel oblique* (Defs. 8, 9, and 11), will if produced meet in xy.

ORTHOGRAPHIC ILLUSTRATIONS.

Fig. 3.

Points.



Examples of points determined in various positions by their projections.

Point *A* or *aa'* is in the 1st dihedral angle of the planes of projection.

„ *B* or *bb'* „ 2nd „ „ „ „ „

„ *C* or *cc'* „ 3rd „ „ „ „ „

„ *D* or *dd'* „ 4th „ „ „ „ „

„ *E* or *ee'* is in the horizontal and behind the vertical plane.

„ *F* or *ff'* is in both planes.

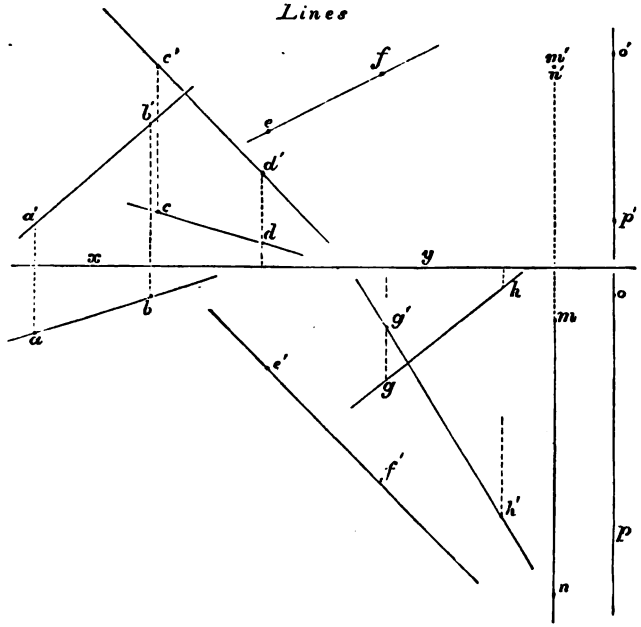
„ *G* or *gg'* is in the vertical and below the horizontal plane.

„ *H* or *hh'* is equidistant from the horizontal and vertical planes.

„ *K* or *kk'* is in the vertical and above the horizontal plane.

Observe. That to find the projections of points satisfying given conditions of position with regard to the planes of projection the relations indicated in Theorems IV. and V. will suffice.

Fig 4.



Examples of lines in various positions determined by their projections.

The segment AB or $ab, a'b$ is in the 1st dihedral angle of planes.

„ CD or $cd, c'd'$ „ 2nd „ „

„ EF or $ef, e'f'$ „ 3rd „ „

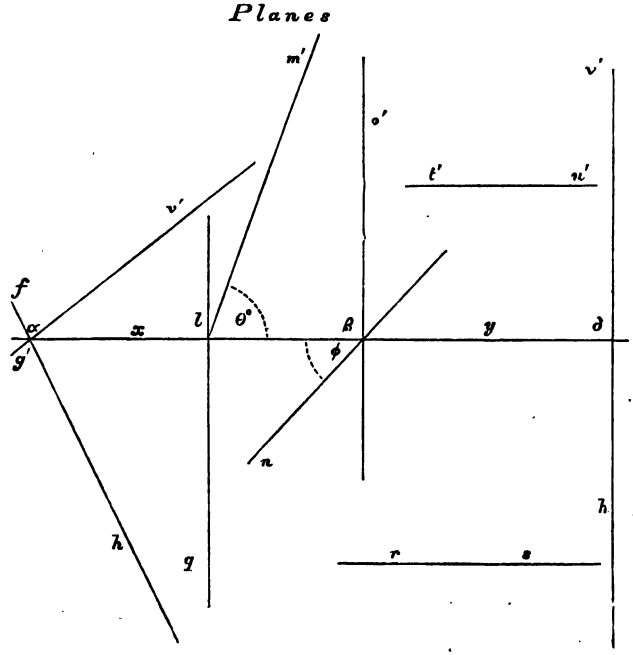
„ GH or $gh, g'h'$ „ 4th „ „

„ MN or $mn, m'n'$ „ 1st „ „

and the line is perpendicular to the vertical plane.

„ OP or $op, o'p'$ is in the 1st dihedral angle.

Fig. 5.



Examples of planes defined or expressed by their traces.

The portion included between the infinite branches ah and av' of the given plane is in the first dihedral angle of the planes of projection.

The portion between the branches av' and af is in the second dihedral angle.

The portion between af and ag' is in the third angle, and the portion between branches ag' and ah is in the fourth angle.

The plane qlm' is perpendicular to the vertical plane and makes an angle of θ^0 with the horizontal.

The plane $n\beta o'$ is at right angles to the horizontal and makes an angle of ϕ^0 with the vertical plane.

The plane rs, $t'u'$ is parallel to xy and meets both planes.

The plane $h\delta v'$ is at right angles to xy and therefore to both planes of projection.

MEMORANDA

FOR WORKING THE PROBLEMS IN ENSUING CHAPTERS.

All the solids given are assumed to be '*right*,' unless otherwise expressed.

All dimensions and arrangements, not expressly mentioned and limited, may be assumed at pleasure.

The inclinations of all lines and planes must be understood, unless otherwise mentioned, to be *to the horizontal plane.*

Invisible edges are those which are hidden by the solid from the eye when looking in the direction in which the *projectors* are drawn.

Sections. The teacher should give directions for sections and developments, as far as possible, with each problem.

Also several *plans* and *elevations* may advantageously be completed of the solids given in many of the problems.

I.

SOLIDS IN SIMPLE POSITIONS.

1. *Draw plan and elevation of a pyramid,* 3·5 *inches high, with square base of* 2·5 *inches side, when resting with its base on the horizontal plane, and with one side of the base making an angle of* 30° *with the vertical plane.*

Commence with the *plan.* This will be the square *ABCD* of the base with the opposite corners joined for the plans of the slant edges of the solid. The point in which the diagonals of the square cross will be the plan *v* of the vertex *V.*

For the *elevation* take the ground line *xy* inclined 30° with one side of the square and draw perpendiculars to *xy* (Theorem IV.) from the four corners *abcd* and the centre *v* of the square. The elevations *a'b'c'd'* of the four corners of the square will be in the ground line, because the base of the solid rests on the horizontal plane. The height of the pyramid, 3·5 inches, must be set up for its vertex *v'* from the point where the perpendicular from the centre of the square meets the ground line, and at right angles to that line (Theorem V.). The whole elevation will be completed by joining the elevation *v'* of the vertex to the four points determined *a'b'c'd'* on the ground line for the elevations of the four corners of the base.

Invisible edges. In this *plan* all the edges of the solid

are visible. In this *elevation* the slant edge of the pyramid nearest *xy* is invisible. The two sides of the base nearest *xy* are also invisible, but as they coincide with the two other sides in the *xy*, they cannot be shown in dotted lines.

NOTE. It must be understood that from any one *plan and elevation* any number of elevations can be drawn by assuming vertical planes, that is, by taking new ground lines, in different positions round the solid, and working from the *plan* to them. For example, if a solid were placed on the floor of a room, four elevations of it might be shown on the four walls of the room, which are so many vertical planes of elevation. And so from any one *elevation and plan* any nnmber of plans may be determined by assuming horizontal planes in the desired position about the elevation, and working from the *elevation* to them.

E.g. *For a new elevation.* The new ground line, *xy*, being taken, draw perpendiculars to it from the *plans* of the various points of the solid, and on these perpendiculars from *xy* mark the various heights of the points, to be taken from the elevation already drawn, and complete by joining the points thus found as already joined in plan.

For a new plan. The new *xy* being taken, draw perpendiculars to it from the *elevations* of the various points of the solid, and on these perpendiculars mark from *xy* the various distances which the points are from the first plane of elevation to the *plan* first drawn, and complete by joining the points as already joined in elevation.

To work a section of the solid. Draw a line through the middle point of the elevation of the axis making an angle of 40° with *xy*. Assume this to be the *vertical trace*

(Def. 6 and Theorem II.) of an ***inclined section plane*** (Defs. 7 and 10). Produce the ***vertical trace*** to meet xy and from the point of intersection of these lines draw the ***horizontal trace*** perpendicular to xy. (Theorem III.).

The points $1'$, $2'$, $3'$, $4'$, in which the given line or vertical trace cuts respectively the lines $a'v'$, $b'v'$, $c'v'$, and $d'v'$ are the *elevations* of the points I., II., III., IV. in which the section plane cuts the edges AV, BV, CV and DV of the pyramid. (Theorem VI. (2) and (α)). The *plans* of these points will be found by drawing lines from their elevations at right angles to xy to cut the corresponding lines in plan. (Theorem IV.). *E.g.* A line at right angles to xy, drawn from the point $1'$ in $a'v'$ to cut the plan av, will give the point 1 which is the plan of the real point I., in the section, of which $1'$ is the elevation. Similarly, the points 2, 3, 4 in the plan may be found. The figure got by joining 1 to 2, 2 to 3, 3 to 4, 4 to 1 is the *plan* of the section. This for clearness should be conventionally crossed with a series of parallel lines. The *elevation* of the section is shown by the points $1'$, $2'$, $3'$, $4'$ in the *vertical trace.* (Theorem VI. (2) and (α)).

To find the true shape of the section.

This is done by the *method of rabatments* or "*construction.*" (Def. 12 and Theorem XX.). That is, the plane of section is rotated about one of its traces as an axis into the plane of projection containing the trace which is chosen for the operation. See Theorem XX.

α. *To find the true shape of the section by rabatting the plane about its vertical trace.*

The vertical projectors (Def. 4) of the points I., II., III.,

IV. of the section, i.e. the perpendiculars from these points to their elevations 1′, 2′, 3′, 4′, lie in the plane of section (Theorem VI. (2)), are at right angles to the vertical trace (Theorem VI.), and are equal in length to the perpendiculars drawn from the plans of the points 1, 2, 3, 4 to *xy*, (Theorem V.)—*i.e.* the *plane of section* is the *projecting plane* of the section. If, therefore, lines be drawn from the points 1′, 2′, 3′, 4′, of the elevation, at right angles to the vertical trace of the section plane, equal in length to the respective distances of the points 1, 2, 3, 4 of the plan from *xy*, and the points I., II., III., IV., so found, at the extremities of the perpendiculars, be joined in order, the figure produced will be that resulting from the rabatment of the plane of section about its vertical trace, *i.e.* will be the *true shape of the section.*

β. To find the true shape by rabatting the plane about its horizontal trace. If the plane be rotated about the H.T. as an axis into the horizontal plane, the points I., II., III., IV. of the section will describe segments of circles about the H.T. in planes perpendicular to it, and the loci of these points in plan will be straight lines drawn from the plans of the points 1, 2, 3, 4 of the section at right angles to the H.T. of the section plane. If on these lines are measured off from H.T. the distances of the corresponding points 1′, 2′, 3′, 4′ *up the vertical trace* from *xy* (*i.e.* the radii of the segments of the circles described by the points of the section when the plane is being *rabatted*), the figure produced by joining these points will be the *true shape of the section.*

NOTE. To avoid the confusion that sometimes results from rabatting the section plane directly, the section may

be projected on a plane taken parallel to the plane of section, and will then be shown in true shape. (Theorem VIII.). This, which is merely the method of rabatments under a disguise, is of little use unless the section plane be perpendicular to one of the planes of projection. The xy for the parallel plane must obviously be taken parallel to the trace of the section plane on that plane of projection to which the section plane is perpendicular. Finish by drawing the projection of the frustum of the pyramid on this plane.

To find the true length and inclination of a slant edge of the pyramid.

α. Make a projection on a plane parallel to the edge whose length is required, *i.e.* take xy parallel to the plan of the slant edge. The elevation will show the inclination and the edge in true length. Theorem VIII. or,

By a special construction,

β. Make a right-angled triangle having its two sides respectively equal to the plan of the edge and the length of the axis—the hypotenuse will give the true length of the edge; and the angle subtended by the axis will be the angle which the edge makes with the horizontal plane.

To work the development of the frustum. See Def. 13.

Describe a circle with radius equal to the slant edge of the pyramid and cut off four consecutive chords of this circle equal to AB, BC, CD, DA, the sides of the base of the pyramid. Draw AB, BC, CD, DA, and join the points A, B, C, D, with the centre V of the circle. Describe a square on one of the bases of the four triangles thus produced. The four triangles and the square together

form the development of the pyramid. Find now the points I., II., III., IV. in the edges *VA*, *VB*, *VC*, *VD*, by dividing each of these edges in the same proportion as the corresponding elevations are divided by the vertical trace of the section plane. Join the points in order (noting that in the development one edge and its point of division necessarily occurs twice) and cut the part next the vertex away. This is the development of the frustum. Build up into a model and fit on the true shape of the section.

2. *Plan and elevation of a pyramid with base a regular hexagon of* 1·75 *in. side, when resting on its base, and with one side of base inclined* 40° *to the vertical plane. Axis* 3·5 *inches.*

Work also a section by a vertical plane [Fig. 5 *nβo'*], the angle ϕ^0 being 35°, and the horizontal trace ·25 in. from the plan of the axis. Show new plan and new elevation of the frustum.

Find true shape of section by rabatting or "constructing" the plane of section about one of its traces, and develop the frustum.

3. *Plan and elevation of a square prism* 3·5 *in. long, bases* 2·5 *in. side.*

(α) *When standing on one of its bases and a rectangular face making an angle of* 30° *with the vertical plane of projection.*

(β) *When resting on a rectangular face and its bases making angles of* 50° *with the vertical plane.*

To work the prism in (α) *with section, true shape, projections of the frustum, and development.*

The plan will be a square with one side inclined at 30° to xy.

Begin by drawing xy. Set off from xy a line at 30° in the H.P. and describe on any segment ab, 2·5 in. long, of this line a square *abcd*. This will be the plan of the prism, *i.e.* of its two bases—one on H.P. and the other 3·5 in. above—and its four vertical faces. The elevations $a'b'c'd'$ of the four corners $ABCD$ of the lower base will be in xy, and the elevations $a_2'b_2'c_2'd_2'$ of the four corners $A_2B_2C_2D_2$ of the upper base will be in a line parallel to xy and 3·5 in. above it. The segments $a'a_2'$, $b'b_2'$, $c'c_2'$, $d'd_2'$ of the lines drawn from the four corners of the plan *abcd*, at right angles to xy for the elevation, will be the elevations of the edges AA_2, BB_2, CC_2, DD_2, respectively. Make the edge AA_2, which is nearest the V.P., dotted in the elevation; and the plan and elevation will be complete.

Section. Take a line cutting the elevations of the upper base $a_2'b_2'c_2'd_2'$, and three of the edges $a'a_2'$, $b'b_2'$, $c'c_2'$, in the points 1′, 2′, 3′, respectively, for the vertical trace of the section plane. The elevation of the section will be in this line. For the plan, draw a line from the point in which the vertical trace cuts the elevation of the upper base, at right angles to xy, and produce it to cut the sides *ad* and *dc* of the plan *abcd* in the points *s* and *z*. The figure *sabcz* will be the *plan* of the section. Find the true shape by *rabatting* the section plane about one of its traces as in Prob. 1.

New elevation of frustum. Take a new xy at right angles to the xy first drawn, and project as for a complete elevation of the prism. Show the elevations of S and Z in the new elevation of the upper base, and find the points 1′, 2′, 3′,

by measuring the heights of these points from xy in the elevation first drawn and setting them off on the corresponding edges in the new elevation. Join up the points of the section for the complete elevation of the frustum.

A new *plan* of the frustum can be made in a similar manner by taking a new xy at, say 30°, with the elevations of the long edges of the prism—using either of the above elevations of the prism.

Development of frustum. Draw two parallel lines 3·5 in. apart and measure off on the lower one the segments AB, BC, CD, DA, equal to the sides of the square base of the prism. From the points A, B, C, D, A, draw lines at right angles to cut the second parallel line in the points A_2, B_2, C_2, D_2, A_2. On DA and D_2A_2 describe the squares $DABC$ and $D_2A_2B_2C_2$. This completes the development of the prism.

For the *frustum* find the points I., II., III. in the edges AA_2, BB_2, CC_2, and the points S and Z in A_2D_2 and D_2C_2. The lengths AI., BII., CIII., can be got from the elevations and the lengths D_2S and D_2Z from the plan. Fit on the true shape of the section and build up the model.

The prism in (β) the student should now be able to work for himself.

4. *Plan and elevation of a right cylinder; axis* 3·5 *in. horizontal and inclined* 40° *to the vertical plane. Diameter of base* 2·5 *in.*

Conceive the cylinder to be inscribed in the prism (β) Prob. 3. The plan of the prism, which is a rectangle 3·5 in. long and 2·5 wide, is also the plan of the cylinder. The *elevation* of the cylinder is made by inscribing ellipses in the

rectangles which are the elevations of the square ends of the prism.

To work a section of the cylinder take an auxiliary *xy* at right angles to the plan of the axis and draw the circle which is, in this position, the elevation of the cylinder. Divide this circle into, say, sixteen equal parts and draw, from each of the points, lines in plan parallel to the axis. These will be the plans of lines lying on the surface of the cylinder. Show these lines on the first elevation of the cylinder by measuring their heights in the auxiliary elevation and transferring them. A *vertical* or *indined* section plane may now be taken and the section determined by finding the projections of the points in which the given plane cuts these lines. The points found must be joined by a curved line put in by hand. The section of a right cylinder by a plane which is neither parallel nor at right angles to its axis, as in this case, is an ellipse. Find the true shape by *rabatting* the plane.

To work the development. Measure a straight line equal to the circumference (found by multiplying the diameter of the cylinder by 3·1416) and divide this line into sixteen equal parts. Draw lines 1, 2, 3, 4, &c. from these points and make each equal to the length from the base to the section plane of the corresponding line on the cylinder.

Join the points so found by a curve for the development of the cylindrical surface.

5. *Right cone with axis* 3·5 *in. and diameter of base* 2·5 *in.*

a. *Plan and elevation when standing on its base, and a second plan when resting on its conical surface.*

β. Plan and elevation when the axis is horizontal, 2 in. above the H.P., and inclined 35° to the V.P.

α. The plan of the cone when standing on its base will be simply a circle 2·5 in. diameter, and its elevation an isosceles triangle, base 2·5 in., and vertex 3·5 in. high. For the second plan *xy* must be taken parallel to one of the slant sides of the isosceles triangle. To find the new plan of the base, circumscribe the base first drawn by a square with one side parallel to *xy*. The new plan of this square will be a rectangle in which an ellipse can be drawn for the new plan of the base of the cone, and the projection completed by drawing tangents from the new plan of the vertex to this ellipse.

β. The projection of the cone in this position presents no special difficulty.

Sections of the cone by inclined and vertical planes may here be worked as in the case of the cylinder by dividing the base of the cone into any number of parts and drawing the generatrices of the cone from these points in the base to the vertex. Treating these generatrices as the edges of a polyhedral pyramid, the projections of their intersections with the section plane may be found, as in Probs. 1 and 2, and joined by a curved line.

Note on the true shape of the sections of a right conic surface. When the section plane cuts all the generatrices, the section is a closed curve and is generally an *ellipse*, the exception being when the plane is at right angles to the axis, in which case the section is a *circle*. A plane parallel to two generatrices cuts an *hyperbola*, the limit of which is the pair of generatrices to which the section plane is parallel.

When the section plane is parallel to only one generatrix the section is a *parabola*, and its limit is the parallel generatrix, in which case the section plane becomes a tangent plane to the cone.

The development of the conic surface will be a segment of a circle with radius equal to a generatrix of the cone, the circular arc of which segment subtends an angle of $\frac{r360}{g}$ degrees at the centre; where r = radius of base and g = length of generatrix.

6. *Plan and elevation of a square prism* 3·5 *in. long, edge of base* 2·5 *in., when its axis is horizontal, inclined at* 45° *to the vertical plane, and its lowest face inclined* 30° *to H.P.*

Assume *xy*. Since the horizontal *axis* is at right angles to the *base* and inclined 45°, the base will be inclined also at 45° to the vertical plane, *i.e.* at the complement of the inclination of the axis. Take, therefore, a line in the H.P. at 45° with *xy* and assume this line as the *xy* of an auxiliary vertical plane at right angles to the axis of the prism. The projections of the two bases of the prism on this plane will coincide and will be shown in true shape. (Theorem VIII.). Draw the square base on this auxiliary vertical plane with one side at 30° with the *xy* and deduce the plan. This plan will be the plan of the square prism with its axis horizontal, inclined 45° to the vertical plane first assumed, and its lowest face inclined at 30°. Complete the elevation on first vertical plane as in other problems.

For the section, assume three points—one in each of three edges of the solid—and apply the principle enunciated

in Theorem VII. to the completion of the plan and elevation of the section made by a plane passing through these points.

7. *Plan and elevation of a prism,* 3·5 *in. long, with bases regular hexagons of* 1·5 *in. side, when its lowest face is inclined* 20°, *its axis horizontal, and inclined* 40° *to the V.P. Section as in the last problem.*

The work is similar to that of the last problem, the auxiliary xy being taken at 50° to the xy first drawn.

8. *Plan and elevation of your instrument case when the lid is open* 60°; *the long edges horizontal and inclined* 30° *to the V.P.*

This is worked in a similar manner to Probs. 6 and 7.

9. *Plan and elevation of a tetrahedron with one face horizontal and* 1 *inch above the horizontal plane. Edge* 3 *inches. One horizontal edge to be inclined* 15° *to the vertical plane.*

The *plan* will be an equilateral triangle of 3 inches side, with the corners joined to the centre for the plans of the slant edges of the solid.

The base being horizontal, its elevation will be a straight line and at the required height, 1 inch, above the ground line.

The edge of the tetrahedron being known, the height of the vertex above the horizontal base can be determined by making a vertical section containing the vertex and one slant edge. To find this, make a right-angled triangle with the length from a corner of the base to its centre for the base of the triangle, and the true length of the edge of the tetrahe-

dron for the hypotenuse. The perpendicular will be the height required.

10. *The surface of a sphere, diameter* 3 *feet, is divided into equal portions by* 4 *great circles, one horizontal and three vertical. Draw an elevation of the sphere and circles on a plane parallel to one of them.*

Work also a section by a plane, perpendicular to V. P., inclined 40°, *and* ·5 *in. from centre of sphere. Scale* $\frac{1}{12}$.

11. *A street-lamp is formed of a right pyramid, and a frustum of one; the slant edges of former inclined* 40°. *The common base of the forms is a square of* 20 *inches side, and the lower base or end of frustum is a square of* 12 *inches side and* 20 *inches below the other.*

(*a*) *Determine an elevation on a plane parallel to a diagonal of the plan.*

(*b*) *Determine an elevation on a plane which makes an angle of* 30° *with a horizontal edge.*

Scale $\frac{1}{8}$.

The *plan* is a square of 2·5 in. with the opposite corners joined for the slant edges, and with a second square of 1·5 in. within and parallel to the first.

In the *elevation* (*a*) the slant edges are drawn at 40° with the elevation of the base of pyramid.

12. *A ladder,* 50 *feet long and of the uniform width of* 3 *feet, rests against a vertical wall—*3 *feet thick,* 50 *feet high,* 20 *feet wide—and makes an angle of* 70° *with the horizontal plane on which it stands.*

(*a*) *Draw a plan, with an elevation on a plane parallel to that of the wall.*

(*b*) *Also an elevation on a plane making an angle of* 50° *with that of the wall.*

Scale 10 *feet to* 1 *inch.*

N.B.—*Rungs and sides to be represented each by a single line. Three rungs only to be shown, dividing the length of the ladder equally.*

Commence with an auxiliary elevation on a vertical plane parallel to the ladder.

For this, draw *xy* and lay off the angle 70°, and on this line measure the length of the ladder; thence determine plan.

13. *A right prism,* 4 *inches long, with square bases of* 2·5 *inches side, is to be drawn*

(*a*) *with one diagonal of the solid vertical,*

(*b*) *with the same diagonal horizontal.*

Draw a plan and elevation of the solid resting on one of the square bases, with one diagonal of the square at right angles to the ground line.

If two opposite corners of the rectangle which is the elevation of the solid be joined, this line will be the elevation of the *diagonal of the solid,* shown from its position in its true length. A plan may then be determined with this diagonal vertical *or* horizontal by taking (for *a*) *xy* at right angles to, *or* (for *b*) parallel with, the elevation of this diagonal.

14. *Plan and elevation of a pyramid with hexagonal base.*

a. With one face ABV *horizontal.*

b. When one face is vertical.

c. *When one edge is horizontal.*

d. *When one edge is vertical.*

Determine plan of the pyramid when resting on its base.

For *a*, Determine elevation on a vertical plane taken at right angles with the plane of face *ABV*. Then the required plan may be deduced by taking *xy* parallel to the line which represents the elevation of face *ABV*, and working to it.

For *b*. Take *xy* perpendicular to the line representing the elevation of face *ABV*.

For *c* and *d*. Determine an elevation in which the edge is shown in its true length. Then *xy* taken parallel for *c*, perpendicular for *d*, to the elevation of this edge will place the pyramid in proper position for the plans required.

15. *Plan and elevation of an octahedron, edge 3 inches.*

(*a*) *When one axis of the solid is vertical and another is inclined at 75° to the vertical plane.*

(*b*) *When resting with one face on the horizontal plane.*

(*a*) Describe a square of 3 inches side and join the diagonals, which will be the plans of the horizontal axes. This square will be a *plan* of the solid in the required position. For the *elevation* it is only necessary to know that the length of the vertical axis is equal to a diagonal of the square, and that the heights of the horizontal edges are half that diagonal. Take *xy* at an angle of 75° with the plan of the diagonal.

(*b*) Work this first in the same way as Prob. 14, (*a*), and then by the following special construction :—

Describe an equilateral triangle of 3 inches side, and from its centre with radius to one of the corners describe a circle. In this circle complete the hexagon, of which the three corners of the triangle are alternate points. This hexagon with the alternate points duly joined is the required *plan*.

For the *elevation* it is necessary to find the distance between the parallel faces. This is done by a vertical section through one axis of the solid. This section is a rhombus, the shorter diagonal of which is an edge of the solid, and the side of which is the altitude of the equilateral triangle forming the horizontal or any other face of the solid. The distance between the opposite sides of the rhombus gives the distance between the parallel faces of the solid.

Note.—It should be observed that this distance is equal to the height of the tetrahedron whose side is equal to that of the given octahedron.

(*c*) *Develop the solid.*

16. *A solid is formed from a cube of* 2 *in. edge by planes which pass through the middle points of its edges and cut off the eight corners.*

(*a*) *Plan when resting on one of the cubic faces; and elevation on a plane taken at* 30° *to one of the vertical faces of the primary cube.*

(*b*) *Plan when a diagonal of the primary cube is vertical.*

17. *A solid is formed from a cube* 1·5 *in. edge by producing a diagonal of the cube to equal distances of* 2·5 *in. on opposite sides, and then taking planes through the extremities of the produced diagonal and the six lateral edges. Show plan and elevation when resting on one face.*

18. *A square, 2 in. side, in the H.P., is the base of an oblique prism; the edges, 4 in. long, are inclined at* 55° *and in plan parallel to a diagonal of the square. Elevation on a vertical plane inclined* 20° *with the plans of the long edges.*

Work also a section and the development of the prism.

19. *An equilateral triangle* abc, *of 3 in. edge, centre* v, *is the plan of a solid, bounded by plane faces, of which the points* A, B, C, *and* V, *are respectively* ·25 *in.,* ·5 *in.,* 1 *in., and* 3·5 *in. high. Determine the elevation on a plane taken at an angle of* 15° *with the line* ab *in plan, and work a section and development of the solid.*

II.

COMBINATIONS AND GROUPS OF SOLIDS.

1. *A prism, with bases equilateral triangles, is placed with one face on the horizontal plane. On the centre of it rests a brick,* 9 *by* 3 *by* 4·5 *inches, with one of the short edges of a large face touching the ground, with the centre of that face touching the highest edge of the prism and with the face itself inclined* 45°. *Horizontal edge of prism* 9 *inches. The side of the equilateral base to be determined from the question. Scale* ½. *Draw plan and elevation, and a second elevation taking the vertical plane at* 30° *with any horizontal edge of the prism.*

Commence with an *elevation* so placed that both bases of the prism coincide. To do this, draw first the elevation of the brick, which will be simply a rectangular parallelogram. Through the lowest corner take *xy* making the required angle with the line that represents the face inclined 45°, and make the centre of this line the vertex of the triangle, recollecting that the elevation of the base of that triangle must according to the conditions be in *xy*, the ground line. The work for the *plan* is simple. The second elevation will be determined by taking the new ground line 30° with any horizontal edge of the prism, and working to it.

2. *Instead of the prism take a cylinder with brick touching it and otherwise disposed as above. The base of the cylinder to be determined from the question;* i.e. *so that the horizontal bisecting line of the face of the brick may be in contact with the cylinder.*

3. *A block,* 3 *in. square and* 1 *in. thick, is pierced by a hexagonal prism* 1 *in. side and* 4 *in. long. Axis of prism passing through centres of square faces of block and projecting equally on each side. Two faces of the prism to be parallel to two narrow faces of the block.*

(*a*) *Plan when the block stands on one of its narrow faces with two faces of the prism vertical: elevation on a vertical plane inclined at* 40° *to the short edges of the horizontal faces of the block.*

Section and true shape by an inclined plane.

(*b*) *Plan when two of the faces of the prism not parallel to those of the block are horizontal: elevation on a plane at* 30° *with the axis of the prism.*

Section and true shape by a vertical plane.

Commence with an auxiliary elevation so placed that the projections of the bases coincide. In this position the elevation of the block will be a square, and of the prism a hexagon, with its centre in the centre of the square and two of its sides parallel to two sides of the square. From this auxiliary elevation both the plans in (*a*) and (*b*) can be determined, and thence the required elevations.

4. *The faces of a tetrahedron* 3″ *edge are the bases of four prisms whose axes are equal in length to half the height of the tetrahedron. Plan and elevation when an axis of the tetrahedron is vertical.*

5. *Draw a line* 5 *in. long in the H.P. parallel to xy.*

From each end of this line cut off segments of 1·5 *in., and describe on them two regular pentagons. Consider these as the bases of two prisms* 2·5 *in. long, and make a plan and elevation when spanned by a pentagonal Gothic arch—the centres of the circles as shown in elevation being the middle points of the elevation of the upper bases.*

6. *A solid is formed of a slab* 1 *in. thick, having regular hexagons of* 2 *in. side for its faces, and an equilateral triangular prism* 4 *in. long passing through the slab and projecting equally from each of its faces—the long edges of the prism containing the alternate short edges of the slab. Plan and elevation when resting on one corner of the slab and one edge of the base of the prism.*

7. *Plan and elevation of a cylinder,* 4 *inches long, diameter of base* 2 *inches, passing through the centre of a circular slab,* 1 *inch thick and* 3 *inches in diameter; the axis of the cylinder at right angles to and extending equally from both faces of the slab. The whole to be placed resting on the rim of the slab and the rim of the cylinder.*

8. *Describe three circles with radii* 1·5 *in., each touching the other two, and join their centres. Consider the circles as the plans of three spheres resting on the horizontal plane, and the triangle as the plan of the base of a tetrahedron resting upon them. Complete plan and elevation of the whole, and draw a second plan when one of the slant edges of the tetrahedron is horizontal.*

For the second plan take *xy* parallel to the elevation of any slant edge of the tetrahedron.

9. *A tetrahedron, edge* 2·5 *inches, stands on a square block, with an edge parallel to a side of the square. The block*

is 1 *inch thick, and its square faces have sides of* 3·5 *inches. An axis of the tetrahedron, if produced, passes through the centres of the squares.*

Draw plan and elevation—

(*a*) *When the square faces of block are horizontal and the plane of elevation makes* 30° *with a side of a square.*

(*b*) *When only one slant edge of tetrahedron is horizontal.*

10. *A circular slab,* 1 *inch thick, diameter* 3 *inches, rests on one face. On it is placed a tetrahedron of* 2 *inches edge, axis vertical and over the centre of the slab. Draw plan and elevation, and a second plan when the rim of the slab and the vertex of the tetrahedron touch the horizontal plane.*

For the second plan take *xy* touching the elevations of the vertex and the rim of the slab.

11. *A solid is formed from a cube of* 3 *in. edge by placing a square pyramid on each of its faces; axis of pyramid equal to half the diagonal of the face. Plan and elevation when one face of the solid is horizontal.*

12. *A regular hexagonal slab* ·75 *in. thick, side of hexagon* 1·5 *in., stands on one of its rectangular faces, the long edges of which make angles of* 40° *with xy. A sphere of* 2·25 *in. diameter is embedded in the slab, the centre of the sphere coinciding with the centre of one of the hexagonal faces. Show plan and elevation.*

13. *An equilateral triangle of* 4 *in. side is the outline of the plan of four tetrahedrons,* 2 *in. edge each, three of which stand upon the H. P.; the fourth rests upon the others with the three corners of its horizontal face coinciding with their summits. Plan and elevation.*

III.

PROBLEMS ON THE STRAIGHT LINE AND PLANE.

Obs. **To express** points, lines and planes, see Problems 1 to 7.

Problem I.

Given the projections ab, a'b' *of a finite straight line to determine:*

1. *The length of the segment* AB *of the given line, or the distance between points* A *and* B.

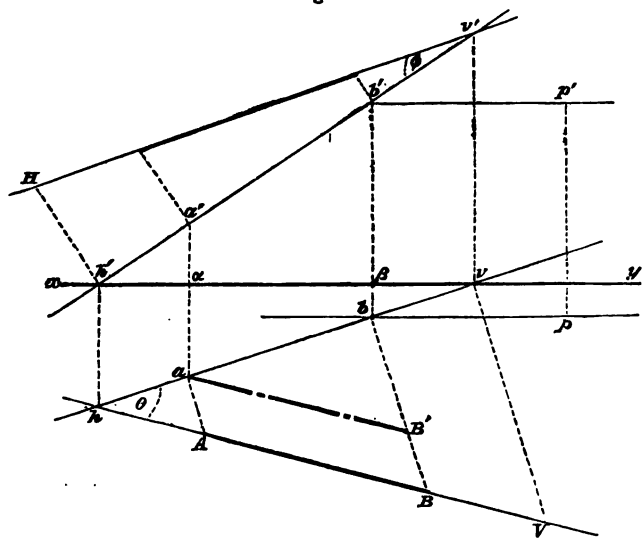
2. *The angle of inclination of the line to each plane of projection.*

3. *The traces of the given line and the distance between them.*

1. The line AB lies in each of its projecting planes: 'construct' or rabat either of these into the corresponding plane of projection and the segment AB is shown.

Let the projecting plane for the plan *ab* revolve about this line as an axis, into the horizontal plane.

Fig. 6.



From points a and b set off the projectors aA and bB, equal to $\alpha a'$ and $\beta b'$ respectively, and at right angles to ab, on the same side of it.

Join A and B; AB is the required distance or length of segment.

Observe. The length AB (being the hypotenuse of the right-angled triangle of which the sides are ab and the difference of the projectors Bb and Aa) may be found by setting this difference from b along bB to B' and joining B' to a.

2. The angle a straight line makes with a given plane is the angle between the line and its orthographic projection on that plane, or the angle which AB makes with ab, and is therefore the angle θ.

For the same reason ϕ is the angle which the given line makes with the vertical plane of projection.

3. The 'traces' of the line being points *in* the planes of projection have each of them a projection in xy, viz. h' and v (Theorem VI.); but the elevation $a'b'$ produced contains h', therefore the intersection of these lines, namely, xy and the elevation, gives point h'; and Theorem IV. and the plan of the line give point h. The corresponding loci of v and v' determine those points.

The distance between the traces H and V of the given line is obviously the length hV, as shown in figure.

Corollaries given below.

As an exercise on this problem determine the inclinations of the given lines, AB, CD, EF, &c., and their traces, and the lengths of the segments indicated. See Fig. 4.

Converse of Problem 1:

(1) Given the traces of a straight line; or

(2) Its inclination and plan; or

(3) Its length and inclination; or

(4) Length and the position or *loci* of the extremities of a finite portion; to draw or determine the projections of the line.

(1) If the traces hh' and vv' be given,—
Join h and v and h' and v'.
Then hv, $h'v'$ are the projections required.

(2) If the plan hv be given making a given angle hva with xy, and the line inclined at an angle of θ degrees.

At any point h in the given plan draw hV, making the angle θ^0 with hv.

Assume points a and b in the plan, draw the perpendiculars aA and bB, and find points a' and b' by Theorems IV. and V. and join a', b'.

Then ab, $a'b'$ are the projections.

(3) If the length AB be given, its inclination θ, and the position aa' of one extremity A.

Through a draw the indefinite plan hv.

Set aA at right angles to ab and equal to aa'.

Through A draw hV, making the angle θ with hv.

From A set off AB along this line and determine b and b'.

ab, $a'b'$ are the required projections.

(4) If the length AB be given and the positions of its extremities A and B with regard to the two planes.

Draw aa' the projections of one extremity A, from the conditions. Theorems IV. and V.

Also any point pp' from the conditions for the other extremity B and through p draw a parallel to xy: this is a locus of b the other extremity of the plan ab required. Determine the length ab* of the plan; measure this length from a to a point b in the parallel, and the position of ab is fixed; b' may be found of the height of p', and ab, $a'b'$ are determined.

The student will observe that the problems admit of one, two, or more, or an infinite number of solutions. These cases should be studied. See Problem 2.

* The base of a right-angled triangle, hypotenuse AB and altitude equal the difference of the heights of A and B.

I. (3), COR. 1. *To determine the 'traces' of a given plane when the data are:*

(1) *Two straight lines (parallel or meeting).*

(2) *A straight line and a point.*

(3) *Three points.*

(1) The horizontal and vertical traces of the given lines must be found by Problem 1, and the corresponding traces of the plane drawn through the points thus found. Theorem X.

(2) Find the traces of the given line.

Draw the projections of a line parallel to it, through those of the given point (Theorem VII.), and find its traces.

Or, by Theorem IV., determine the projections of any point qq' *in* the given straight line; join the projections of this point with pp' those of the given one; then the line PQ lies in the given plane.

The traces of the two lines give those of the plane.

(3) Join the corresponding projections of the given points; and determine the traces of the three lines of which those joining lines are the projections.

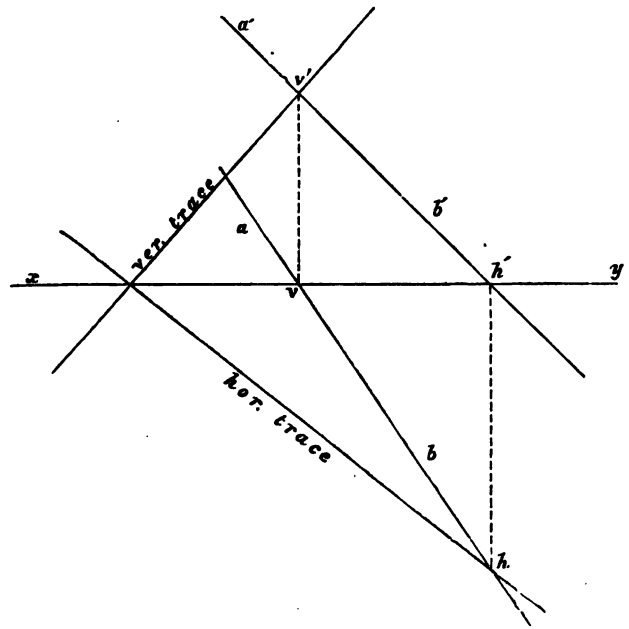
The traces of the three lines give those of the plane.

Obs. If the given plane meets xy only three traces of the lines lying in it need be found; e.g. two in the horizontal trace of the plane and one in its vertical trace: the latter joined to the point of intersection of the horizontal trace of the plane with xy gives the vertical trace of the plane.

I. COR. 2 (see figure). *Given the traces of a plane and one projection* ab *of a straight line which lies in that plane; to determine the second projection* a'b' *of the line.*

Since the traces hh' and vv' of the line must be *in* those of the given plane (Theorem X.), determine these points, then the line drawn through points h', v' is $a'b'$ the required elevation.

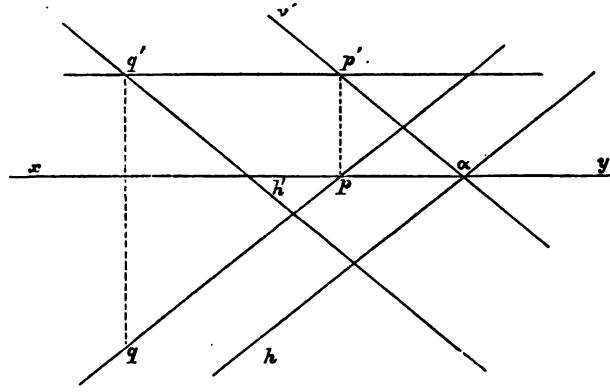
Fig. 7.



Observe. Point H lies in the given straight line AB, hence the projections hh' of the point are in those of the line. Thus ab is a locus of h, and of v; the horizontal trace of the plane is a second locus of h, and xy of v, which points are thus determined: perpendiculars to xy through them are loci of h' and v', Theorem IV., which points are thus also determined.

I. Cor. 3. *To determine a point* pp′ *of given altitude* a *inches* (*or* a″) *in a given plane* hαv′.

Fig. 8.



In the given vertical trace $\alpha v'$, take a point p', a'' high, Theorem V.; find p in xy, Theorem IV.; then pp' is the required point.

I. Cor. 4. *To determine a horizontal line* a″ *high, in a given plane* hαv′. Fig. 8.

Determine pp'' as before.

Through p draw pq parallel to hα, Theorem VII.

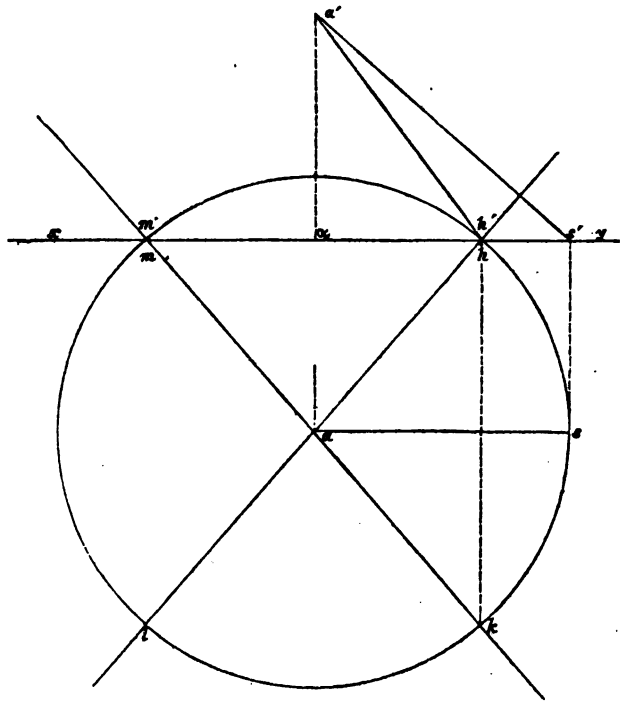
Through p' draw $p'q'$ parallel to xy.

Then pq, $p'q'$ is the required line.

PROBLEM II.

Through a given point aa' *to draw a straight line inclined* θ° *to the horizontal plane.*

Fig. 9.



If a right cone be determined having its vertex in the given point A, axis equal to $a'\alpha$ and generatrix inclined θ, every generatrix of the cone fulfils the given

conditions. The conic surface is therefore a locus of the required line.

To work the problem:

Through point a' draw $a's'$ making the angle $a's'a$ equal to θ.

Through a draw as parallel to xy, then as, $a's'$ is parallel to the vertical plane and inclined θ^0.

With centre a and radius equal to as' describe the circular base klm of the cone which gives the complete solution.

Observe. The four lines AH, AK, AL, AM are inclined θ^0, and make equal angles with the vertical plane.

Problem III.

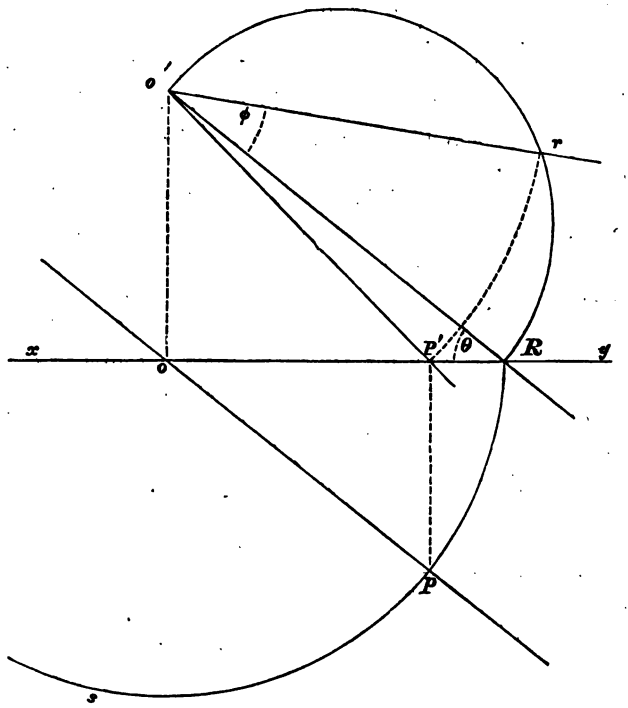
To determine a straight line which shall make a given angle θ with the horizontal plane and an angle ϕ with the vertical plane of projection.

A line inclined θ^0 will lie on the surface of a right cone with axis vertical and generatrix inclined θ^0 to the base.

Determine such a cone having its vertex in point oo' in the plane of elevation: Rps will be its circular base. That generatrix of the cone which makes an angle ϕ with the vertical plane will be the line required. To find this. Knowing the length $o'R$ of this generatrix, determine the length $o'r$ of its elevation as in the figure; and since the position of o' one extremity of it is fixed, that of the other, viz. p', in xy, may be found as indicated. The perpendicular to xy from p' is a locus of p, and intersects the cir-

cular arc Rps, which is another locus of p, which point is thus determined. Join op, $o'p'$; it is the required line.

Fig. 10.



It is obvious that there are four such lines on the surface of the cone, and therefore there are four solutions of the problem, when oo' is a given point in the line.

It will also appear from the above, that the second angle ϕ cannot exceed the complement of θ or $(90-\theta)$ degrees.

Problem IV.

To draw or determine the traces of a plane inclined θ degrees and perpendicular to the vertical plane of projection. See Fig. 5, 'Planes.'

Draw the horizontal trace *ql* of the required plane at right angles to *xy*, meeting it in point *l*; and through *l* draw *lm'* making the angle *ylm'* equal *θ*: *qlm'* is the required plane.

It is obvious that *lm'* and *ly* are the two perpendiculars to *ql*, the horizontal trace, or common section of the planes, at the point *l* in it, which form the measuring-angle of the inclination of these planes to each other.

The plane of *lm'* and *ly* having been turned into the horizontal plane, *lm'* in Fig. is no longer perpendicular to *ql*.

Similarly (see Fig. 5, 'Planes') a plane *nβo'* may be drawn at right angles to the horizontal and making an angle *φ* with the vertical plane of projection.

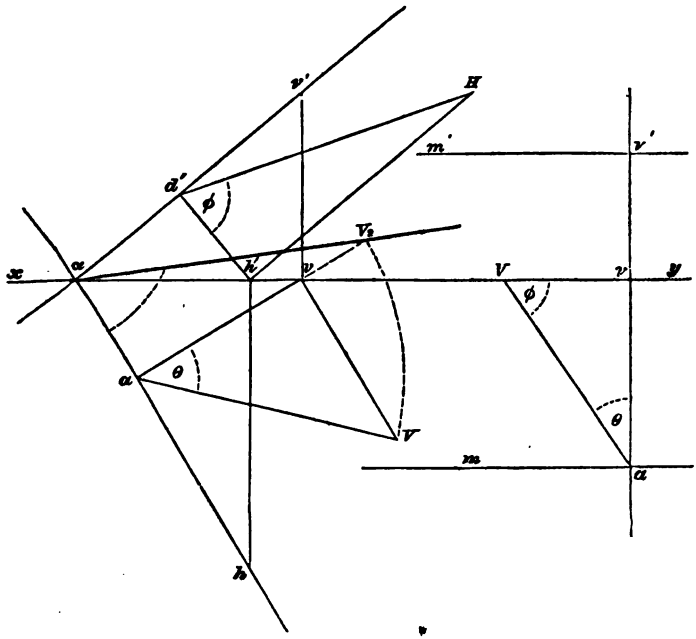
Problem V.

To determine the angles θ and φ which a given plane hav' *or* am, m'v' *makes with the planes of projection; and to determine the angle between its traces.*

Since a dihedral angle is measured by the plane angle formed by two perpendiculars drawn one in each plane to the same point in their common section, the plane of this rectilineal angle is perpendicular to the 'trace' of the given plane. 'Rabat' the plane of this angle, revolving it about that line of it which is *in* the plane of projection in question, and the required angle of inclination is obtained.

Thus at point *a* in the horizontal trace *ha* (Fig. 11) two such perpendiculars *av* and *aV* were found, and at

Fig. 11.



point *d'* in the vertical trace perpendiculars *d'h'* and *d'H*, and the required angles shown.

To determine the magnitude of the angle *hav'* between the traces of the plane.

Let the plane be 'constructed' or rabatted about its horizontal trace *ha* until it coincides with the horizontal plane.

Then the length aV measured from a along av produced to V_2 gives a second point in the second line of the required angle. Join aV_2, and haV_2 is the angle between the traces of the plane.

PROBLEM VI.

Given the horizontal trace ah *of a plane inclined* θ *to determine its vertical trace* av'.

At any point a in the horizontal trace draw av at right angles to it and meeting xy in v. Fig. 11. Draw aV making the angle vaV equal to the given angle θ. From v draw vV at right angles to av, and from v set off the length vv' at right angles to xy and equal to vV. Then the straight line through a and v' is the vertical trace required.

PROBLEM VII.

To determine the traces of a plane making an angle of θ *degrees with the horizontal and an angle of* ϕ *degrees with the vertical plane.*

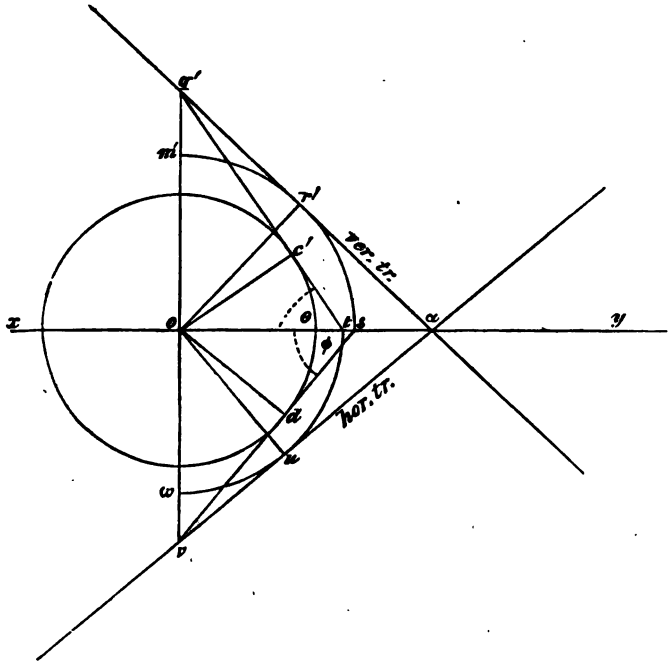
When a plane touches a right cone it is inclined at the angle which the generatrix of the cone makes with its base.

Determine two right cones (Fig. 12) with vertices q' and v and bases tuw and $sr'm'$ in the horizontal and vertical planes respectively; the generatrix $q't$ making an angle θ with its base tuw, and generatrix vs an angle ϕ with its base $sr'm'$. The common tangent plane to these cones is the plane required.

The surfaces of the cones have a common point pp which is also on the surface of a common sphere enveloped

by them and having its centre *o* in *xy*. See Figure 13. Hence the plane of points *P*, *Q*, and *V* is that required.

Fig. 12.

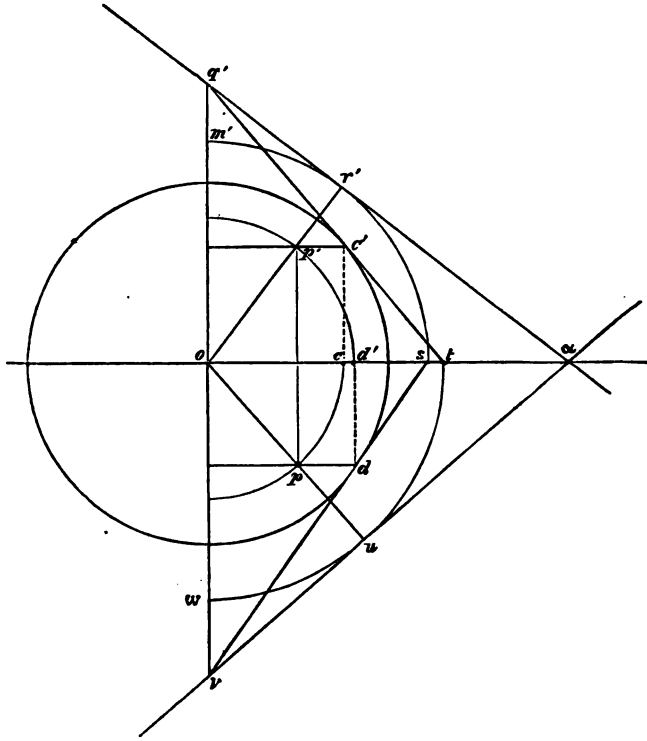


To work the problem.

Commence with one of the vertices, assuming point *Q* or *q'* in the vertical plane. Fig. 12. Draw *q'o* indefinitely at right angles to *xy*, meeting it in *o*; and draw *q't* as above described, making angle θ with *xy*, on which let fall the perpendicular *oc'* from *o*; the circle described with centre *o* and radius *oc'* gives the common inscribed sphere of the

two cones. *vs* drawn as above mentioned, making the angle ϕ with *xy*, and touching this circle in *d*, meets *q'o* produced in the point *v*, which is the second vertex. Tangents to the bases of these cones through *v* and *q'* meet *xy* in point *a* and give the traces of the required plane *q'av*.

Fig. 13.



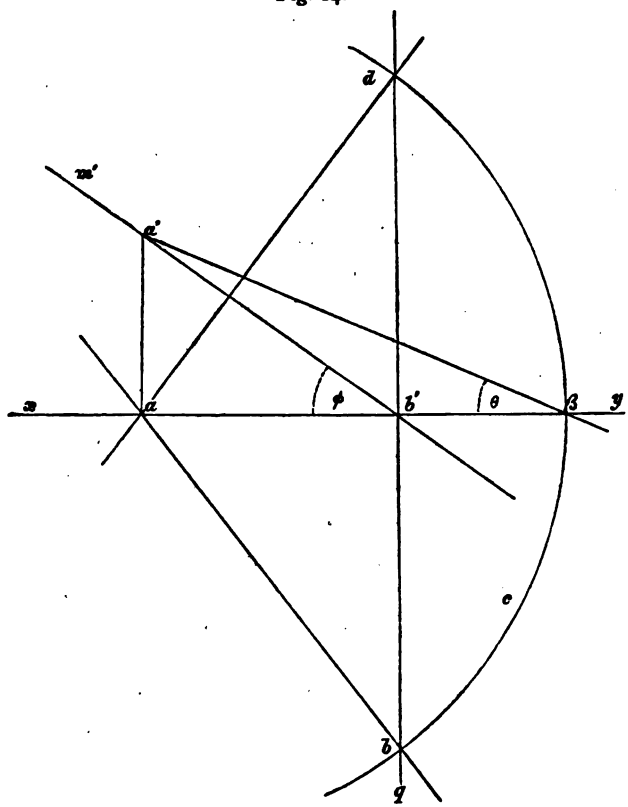
Limits. The sum of the given inclinations or $\theta + \phi$ lies between 90^0 and 180^0.

Obs. **To express** given combinations of points, lines, and planes, see Problems 8 to 28.

PROBLEM VIII.

To determine a straight line inclined θ^0 to lie in a given plane qb'm'.

Fig. 14.



Limits. The angle ϕ is the angle of inclination of the given plane. It is therefore obvious that the angle of inclination θ of the given line must be *not greater than* ϕ. For no straight line can lie in a plane and have a greater inclination than that of the plane.

Construction. Assume point aa' in the given plane.

Make the angle $a'\beta a$ equal θ.

With centre a and radius $a\beta$ describe the arc βcb meeting the trace qb' in b or d; then ab', $a'b'$ or ad, $a'd'$ is the required line.

Observe, that each of the lines AB and AD satisfies the conditions of the problem.

The cone, vertex A and base dcb, is the locus of all lines through A inclined θ.

The given plane is a second locus of the required line. And these loci intersect in the lines AB and AD only.

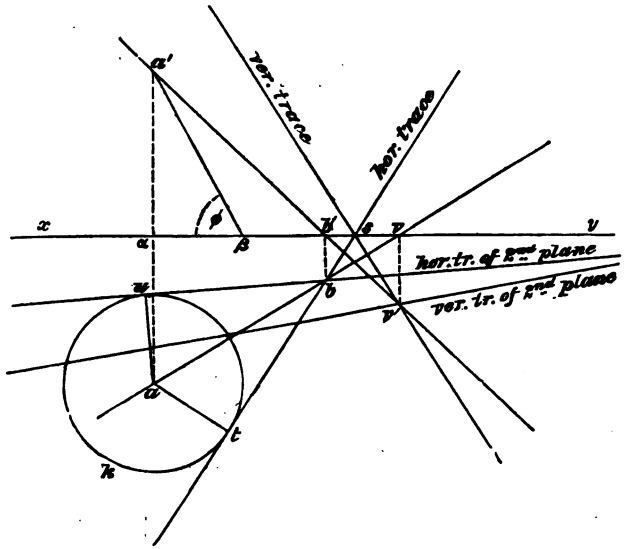
PROBLEM IX (*converse of Problem VIII*).

Through the given straight line ab, a'b' (*which is inclined* θ) *to draw a plane inclined* ϕ^0.

Since the given line is to lie in the required plane, its traces must lie in those of the plane, Theorem X. Determine these traces bb', vv'. Problem I.

Determine a right cone, with vertex in point aa' of the given line, and base *tuk* in the horizontal plane, its generatrix making the given angle ϕ with the base. Every tangent plane to this cone is inclined ϕ^0, Theorem XI.; and the straight line through b touching the base of the

Fig. 15.



cone in t is the horizontal trace of such plane, Theorem XII. By producing this line to meet xy in s and joining sv', we obtain the vertical trace of the required plane.

For this plane tsv', passing through points A and B in the given line, contains that line; and touching the cone has the required inclination.

Note: That the radius at of the base of the cone is taken equal to $a\beta$ in figure.

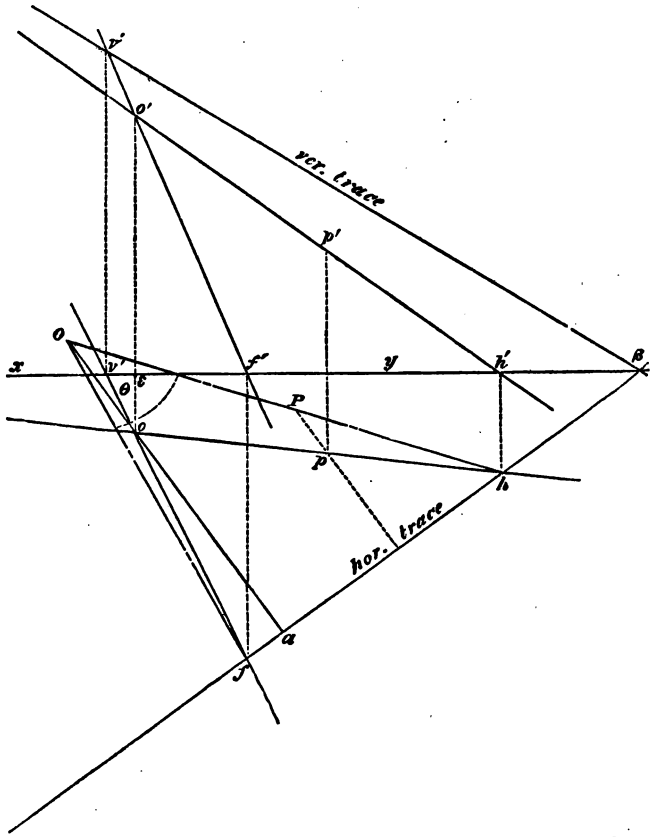
Also, that a second plane bwv', having ub for its horizontal trace, likewise satisfies the conditions of the problem.

Generally, two planes are possible as long as ϕ^0 exceeds θ^0; when ϕ^0 is equal to θ^0 only one plane can be drawn, and when ϕ^0 is less than θ^0 the problem is impossible.

Problem X.

To determine the magnitude of the angle which the two given straight lines OH, OF *make with each other.*

Fig. 16.



Determine the horizontal traces ff' and hh' of the given lines, then (Theorem X.) the horizontal trace $f\beta$ of the

plane of the lines passes through points f and h, and its vertical trace $\beta v'$ through v', vv' being the vertical trace of OF.

'Rabat' the plane $f\beta v'$, revolving it about its trace $f\beta$ into the horizontal plane; then fOh the required angle is exhibited.

The indefinite perpendicular on the horizontal trace through the point o is a locus of O (Theorem XX.), and the length Oa is made equal to the hypotenuse of the right-angled triangle, of which oa is the base and $o'\epsilon$ the altitude.

COR. *To bisect the angle formed by two given straight lines, as* OF *and* OH.

'Rabat' the plane of the angle as above; and bisect the angle fOh thus found by a straight line Os meeting the horizontal trace used in point s; obtain s' in xy, and join os, $o's'$; these are the projections of the required bisecting line.

PROBLEM XI. (See Fig. 16.)

To draw a straight line oh, o'h' *through a given point* pp' *to make a given angle* θ *with a given straight line* of, o'f'.

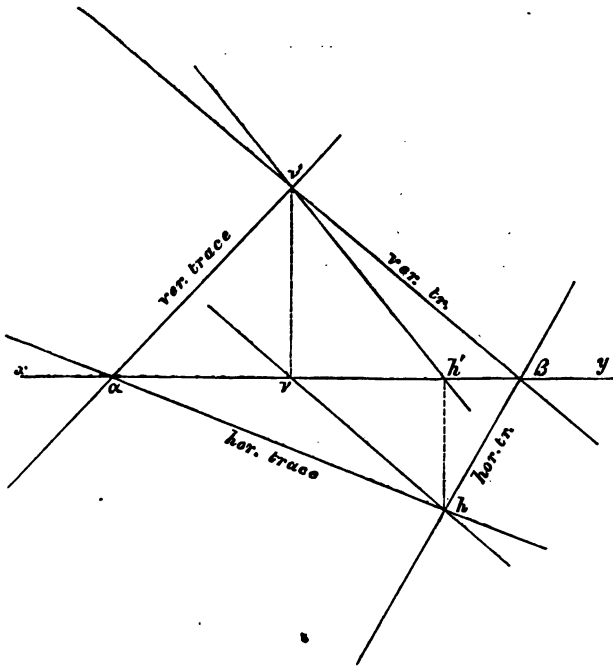
Determine the horizontal trace $f\beta$ of the plane of the given straight line OF and point P (Prob. 1, Cor. 1). 'Rabat' this plane, and through P so 'rabatted,' draw the line OP making the given angle θ with OF in the horizontal plane. Determine the projections oo' of point O, join op, $o'p'$, they are the projections of the required line.

Note: That the construction of this problem is made more simple by using a second plane of elevation perpendicular to that of the angle. Theorem VI. (2).

Problem XII.

To determine the projections of the intersection of two given planes hαv′ *and* hβv′.

Fig. 17.



The planes intersect in a straight line, which lying in both planes has its traces in those of the planes. Hence v', in which the vertical traces meet, is the vertical trace of the line of intersection, the *plan* v of this point is in xy, and is found by a perpendicular to it (Theorem IV.). Simi-

larly, h is the horizontal trace of the line required, and h', its elevation, is found in xy by a perpendicular to that line.

Join hv, $h'v'$, these are the projections of the required line.

COR. 1. *To determine the intersection of three given planes.*

Determine the intersection HV as in the problem, of planes (1) and (2): then the intersection AB of planes (2) and (3). AB will meet HV in a point P which is the intersection required and is the vertex of the solid angle formed by the three given planes.

COR. 2. *To determine a straight line which shall pass through a given point* oo' *and meet two given straight lines* ab, a'b', *and* de, d'e'.

Determine the traces of the plane of the point O and the line AB; likewise those of the plane of point O and the straight line DE. Then the intersection of these two planes is the line required.

Or, find the intersection of one of the given lines with the plane containing the other and the given point O. The line drawn from this point of intersection to O will be the line required.

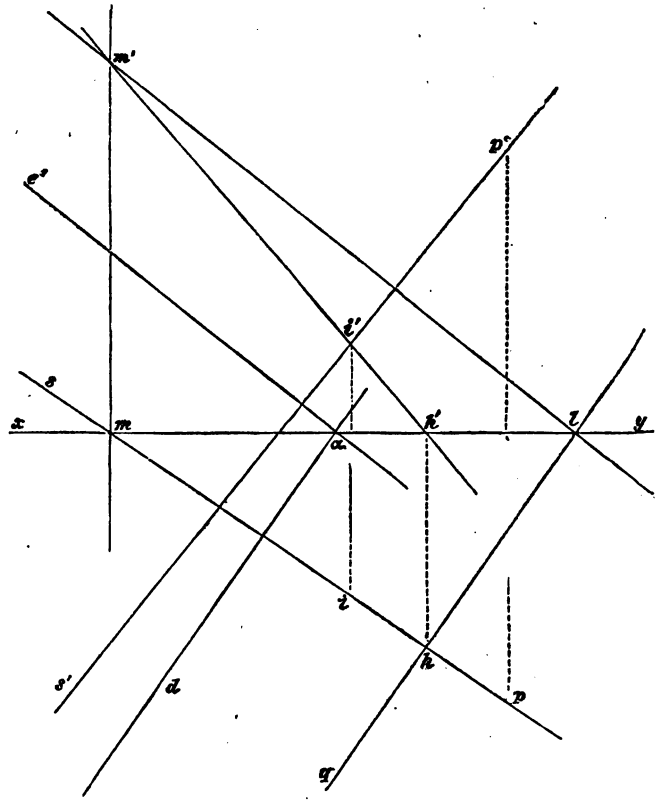
PROBLEM XIII.

To determine a straight line at right angles to a given plane qlm' *which shall pass through a given point* pp'. Fig. 18.

Since the projections of a perpendicular to a plane are perpendiculars to the 'traces' of the plane, Theorem IX.; ps drawn through a point p at right angles to ql, and $p's'$ at right angles to lm', are the projections of the required line.

Conversely. To determine a plane at right angles to a given straight line *ps*, *p's'* which shall pass through a given point *ii'* in the same.

Fig. 18.



Draw the traces *da*, *e'a* of any plane *dae'*, by the problem, at right angles to the given line.

Determine the projections of any line lying in this plane and through the given point draw a parallel *ih*, *i'h'*, to the line determined, and find its trace *hh'*.

Through *h* draw *ql* at right angles to *ps* meeting *xy* in *l*, and through *l* draw *lm'* at right angles to *p's'* : *qlm'* is the required plane.

COR. 1. *To determine the distance of a given point* pp' *from a given plane* qlm'.

Before reading this corollary study the following Problem No. 14, and by it determine point *ii'* in which the perpendicular *PS* to the plane intersects it.

Then by Problem 1 determine the length *IP* of the perpendicular of which *ip*, *i'p'* are the projections. *IP* is the required distance of the given point *P* from the plane.

COR. 2. *To determine two parallel planes at a given distance* (n) *inches from each other.*

Determine a straight line *BC*, and in it mark off the length *PQ*, points *P* and *Q* being (*n*) inches apart; the planes through these points at right angles to the line *BC* are those required.

PROBLEM XIV.

To determine the point of intersection of the given straight line ps, p's', *with the given plane.* Fig. 18.

Every plane which contains the given line meets the given plane in a straight line which passes through the required point of intersection of the given straight line and plane. Assume any such plane *pmm'*, of which *pm* and *mm'* are the traces.

Determine the line of intersection *hm*, *h'm'* of this assumed plane with the given one.

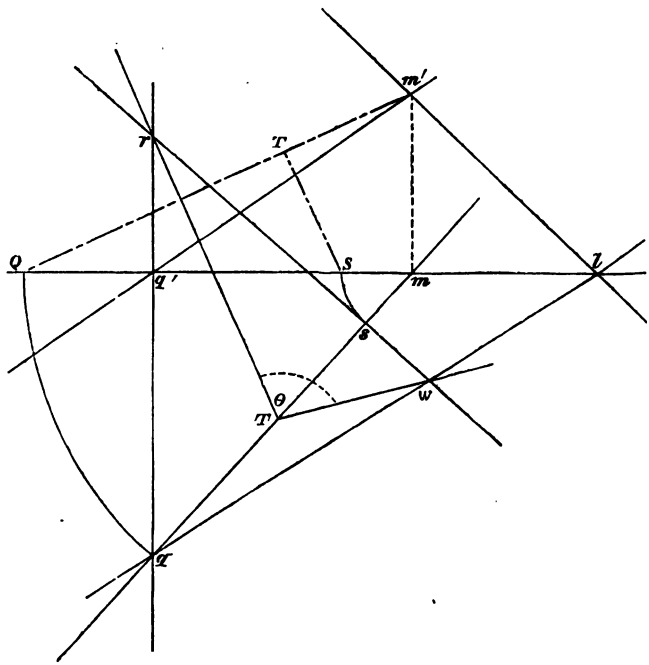
The projections *ii'* of the intersection of these two lines *HM* and *PS* are those of the required point.

Another mode of solving this problem is to make an elevation of the given line and plane on a plane at right angles to the latter: point *i'* in this elevation will be the intersection of the elevation of the line with the vertical trace of the plane: whence the plan *i* may be obtained.

Problem XV.

To determine the angle which the two given planes qlm', qq'm' *make with each other*, i. e. *to determine their 'profile' or 'measuring angle.'*

Fig. 19.



This plane-angle is formed by two straight lines drawn one in each plane from a point in their common section QM at right angles to that line. (Euc. Bk. XI. def. 6.) The plane of these perpendiculars is therefore (Euc. Bk. XI. 4) at right angles to the line QM: hence if this plane be determined and 'rabatted' about its horizontal trace, the required angle wTr is shown.

To do this, assume the horizontal trace rw of the plane of required angle, at right angles to qm, meeting it in s, and the horizontal traces of the given planes in r and w.

Fold the plane about rw bringing the point T in QM down into the horizontal plane in qm: join Tr, Tw.

The distance Ts is found by making it equal to Ts on the vertical plane (as shown), this line being the common section of the 'projecting plane' of the *plan* qm with the plane of the measuring angle.

Conversely. Given θ the profile or measuring angle of the inclination of two planes to each other, to determine the traces of the planes.

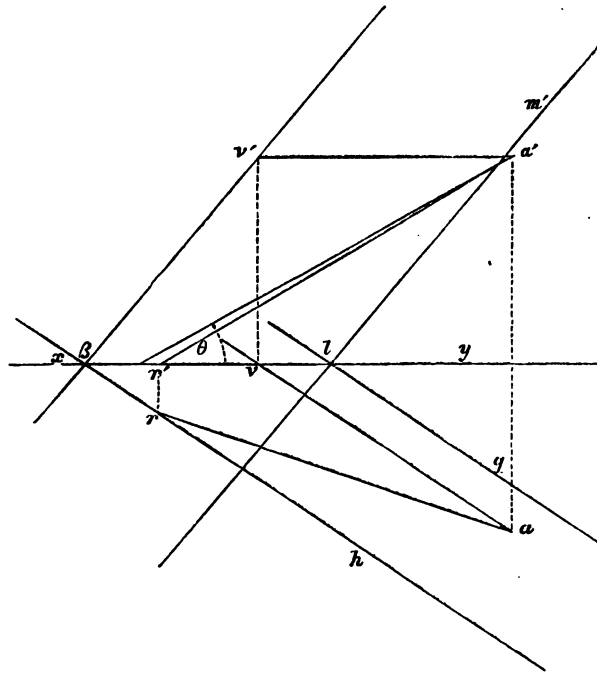
Draw the plane $qq'm'$ and determine the straight line qm, $q'm'$ in it.

Draw rs at right angles to qm and determine sT from ST as before and join rT: then make the angle $rTw = \theta$, and point w is determined, through which draw ql the horizontal trace of the second plane: lm' its vertical trace is drawn through the vertical trace of the line QM (Theorem X.).

PROBLEM XVI.

Through a given point aa′ *to draw a plane parallel to a given plane* qlm′.

Fig. 20.



Since if two parallel planes be cut by a third plane, their sections with the third are parallel straight lines (Euc. XI. 16), therefore the traces of the given and required planes will be parallel straight lines.

If therefore the given plane be not parallel to xy, one point in either of the required traces being found both those lines can be obtained.

Through the given point A let a straight line be drawn parallel to either trace of the given plane; the trace of this line supplies the required point.

Therefore through A or aa' draw the straight line AV or av, $a'v'$ parallel to ql which is a horizontal line; the trace vv' of this line gives the point v' in the vertical trace $\beta v'$ of the required plane: and βh drawn parallel to ql through β gives the horizontal trace.

PROBLEM XVII.

Through a given point aa′ *to draw a straight line which shall be parallel to a given plane* qlm′ *and have a given inclination* θ. Fig. 20.

Obs. The angle θ cannot exceed the angle of inclination of the given plane.

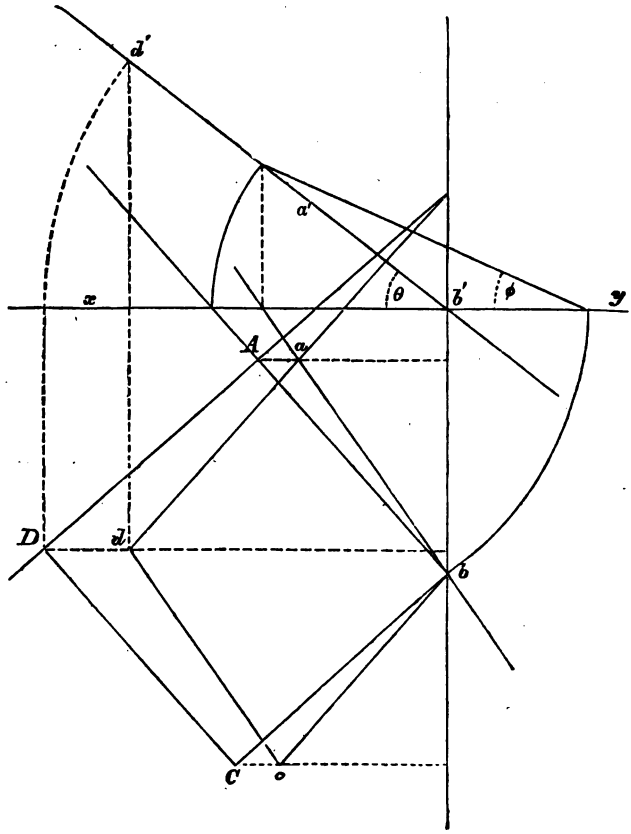
Through the given point aa' let a plane $h\beta v'$ be drawn as in Problem 16, parallel to the given plane. And through point A draw the straight line ar, $a'r'$, by Problem 8, to lie in the plane $h\beta v'$, and have an inclination of θ degrees.

Another mode of solving the problem is, to determine in the given plane a straight line having the required inclination, and to draw through the given point A a straight line parallel to the line so formed.

Problem XVIII.

In a given plane bb'd' *to determine a straight line which shall be perpendicular to a given straight line* ba, b'a' *lying in that plane.*

Fig. 21.



'Rabat' the given plane, revolving it about its horizontal trace bb' into the horizontal plane. Then AB will lie in the horizontal plane in the position Ab. Fig. 21.

The point b (or B) does not move during the rotation of the plane, being *in* the axis or trace bb' of the rotating plane. The perpendicular to bb' through point a is a locus of A, and the position of point A *in it* is found by taking the length $b'a'$ along the vertical trace $b'd'$ of the plane and setting it off from the line bb'. A and b joined gives the line Ab or AB 'rabatted.'

Through point A draw the indefinite line AD perpendicular to Ab, and determine the plan d of any point D in this perpendicular. To do this:—Through D draw a line at right angles to bb' for one locus of d (Theorem XX.). Measure the distance of point D from the line bb' and set this length from point b', along the vertical trace of the plane, to determine d'. A perpendicular from d' to xy is a second locus of d. The intersection of these loci of d determine that point. Join a and d, then ad, $a'd'$ are the projections of the required perpendicular.

Observe. In the same manner a straight line AD may be drawn in a given plane to make any given angle θ with a given straight line AB in that plane.

N.B. Theorems VI. and VIII. should be studied in connexion with this problem and figure.

Problem XIX.

To determine the projections of a given plane figure, when the inclination of its plane is given and that of a straight line lying in the plane of the figure. Fig. 21.

1st. When the given line forms part of the perimeter of the figure.

Let the given figure be a square, its plane inclined θ degrees and one side AB of the figure inclined ϕ degrees.

Determine the plane $bb'd'$: in it place the line ab, $a'b'$. 'Rabat' the plane as in Problem 18, and on AB describe the square $ABCD$: revolve the plane back into its former position and obtain $abcd$, the plan of the figure.

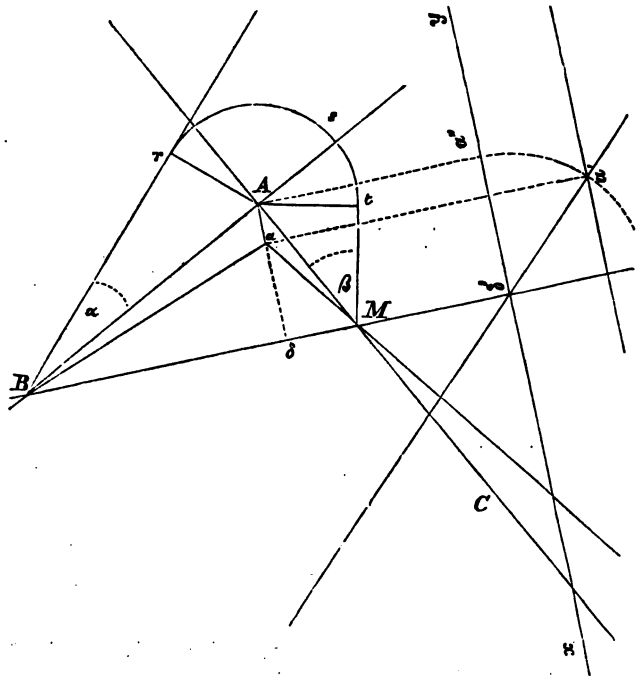
2ndly. When the given straight line is a diagonal of the given figure or its bisecting line: a tangent, a chord, or a diameter of a given ellipse or circle.

Determine the plane $bb'd'$ of the figure and the projection of the given line in it. 'Rabat' the plane, and the line as lying in it; draw the given figure in the horizontal plane having the required position with regard to the 'rabatted' line: and from thence proceed as before to obtain the projections of the entire figure when turned back into the inclined position.

PROBLEM XX.

Given two straight lines AB, AC *making an angle* BAC *of* θ *degrees with each other; to determine the projections of the lines when* AB *is inclined at an angle of* α *degrees and* AC *inclined* β.

Fig. 22.



Observe the conditions of the problem are impossible if $\alpha + \beta + \theta$ exceed 180°:

if $\alpha + \beta + \theta$ equal 180°

the plans of AB and AC will be in one straight line, *i.e.* the plane of the angle BAC will be a perpendicular one.

The case in which $\alpha + \beta + \theta$ is less than 180° will be that here considered.

Draw the angle BAC on the horizontal plane. It will be necessary to revolve the plane of the angle BAC into the inclined position it will occupy when its containing lines have the required inclinations. For this purpose a horizontal line or trace BM of the plane must be discovered, about which the plane is to revolve. Assume a point B in AB and

make the angle $ABr = \alpha°$ and

... ... $ArB = 90°$.

Then Br will be the length of the plan of BA when BA is in the required position.

Now for the same height $Ar\,(=At)$, by the aid of a tangent tM to the arc rst determine

the angle $AMt = \beta°$ and

... ... $AtM = 90°$: and point M is found.

The straight line BM is the required horizontal trace of the given angle. A plane of elevation having its xy at right angles to this line will be always perpendicular to the plane of that angle; and the elevation of the angle upon it will be in the vertical trace, Theorem VI. 2.

The elevation of the path of point A is the circular arc $a''a'$, in which a' is determined by the parallel to xy at the distance Ar.

From a' the plan a is found in $A\delta$ (Theorem IV.).

Then BaM is the required plan, and $\delta a'$ the elevation and vertical trace of the plane of the given angle. $a''\delta a'$ is the angle of the inclination of the plane of the lines.

COR. *To determine three straight lines meeting in a point, each line perpendicular to the other two.*

Proceed, as in this problem, to obtain the projections of the two lines AB and AC; and through point aa', by Problem 13, determine the projections ae, ae' of the third line, at right angles to the plane $M\delta a'$. AB, AC, AE are the required lines.

PROBLEM XXI.

To determine the projection of a given plane figure when the heights of three points in the plane of the figure are given.

Limits. When the given heights are equal, the projection of the figure is an equal and similar figure to the given one.

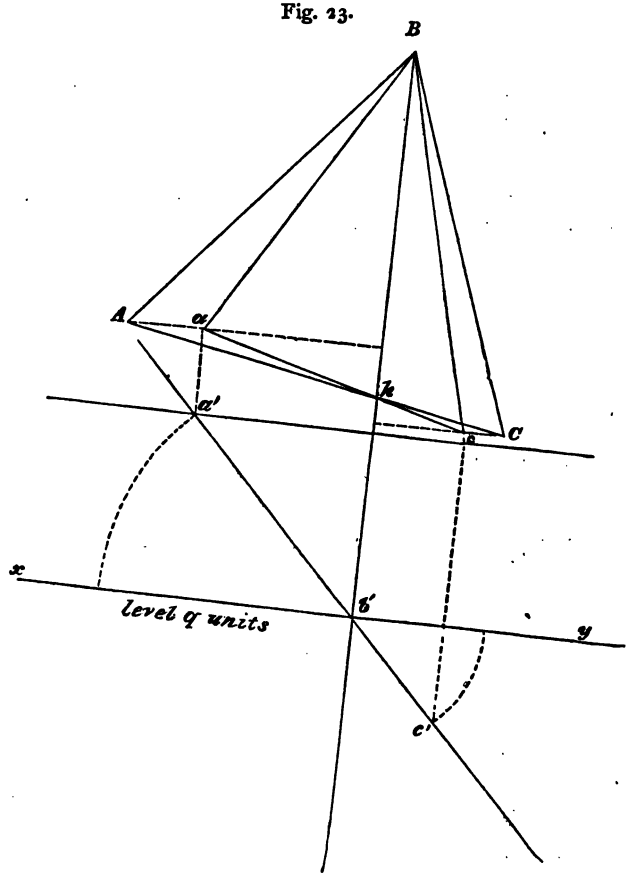
When the given heights are unequal, the problem is a possible one only when the difference between the heights of any two points does not exceed the distance between these points.

If this difference is equal to that distance the plane of the figure is vertical; and the projection required will be a straight line.

Let the three given points A, B and C be at the angles of an equilateral triangle of m inches side, and the heights of A, B and C be p, q and r inches respectively: to determine the plan abc of the triangle. Fig. 23.

(1) Draw ABC in the horizontal plane.

Fig. 23.



(2) Determine point h in AC which will have the same height as B, viz. q inches. Then hB is a horizontal line when the triangle is in its inclined position.

(3) By the aid of an elevation at right angles to hB as shown, the plan abc is determined.

Problem XXII.

To determine two parallel planes which shall contain the given straight lines AB *or* ab, a'b' *and* CD *or* cd, c'd'.

1st. If the given lines *AB* and *CD* are parallel.

Determine the horizontal traces of the lines; parallel lines through these will be the horizontal traces of the parallel planes required, whence the vertical traces can be obtained.

2ndly. If the given lines *AB* and *CD* be neither parallel nor in the same plane.

Through any point *A* in *AB* draw a straight line *AQ* parallel to *CD*.

Determine the horizontal traces of the lines *AB* and *AQ*; the straight line drawn through these points is the horizontal trace of one of the required planes.

Determine the horizontal trace of *CD*; through this point a parallel line to the trace just found is the horizontal trace of the second plane required: and the vertical traces of the two planes can be obtained.

3rdly. If the given lines be not parallel but in the same plane, the problem is impossible.

Problem XXIII.

To determine a straight line at right angles to each of two given straight lines AB *and* CD.

1st. If the given lines be parallel.

Any straight line drawn in the plane of the parallels perpendicular to one of the lines is also perpendicular to the other.

2ndly. When the given lines are neither parallel nor in the same plane.

From any point P in AB draw a line PQ parallel to CD: then the plane of AB and PQ is parallel to CD.

From any point C in CD draw CS perpendicular to the plane of APQ, meeting it in S. A parallel to PQ or CD through point S meets AB in H. Then HT, parallel to CS from H, meets CD in point T and is the required perpendicular to the given lines.

PROBLEM XXIV.

To determine the angle of inclination θ of a given straight line ab, a'b' *to a given plane* qlm'. Fig. 24.

In this problem it is convenient to determine the complement of the angle required.

This complementary angle is the angle between the given straight line and a line meeting it in point A and perpendicular to the given plane.

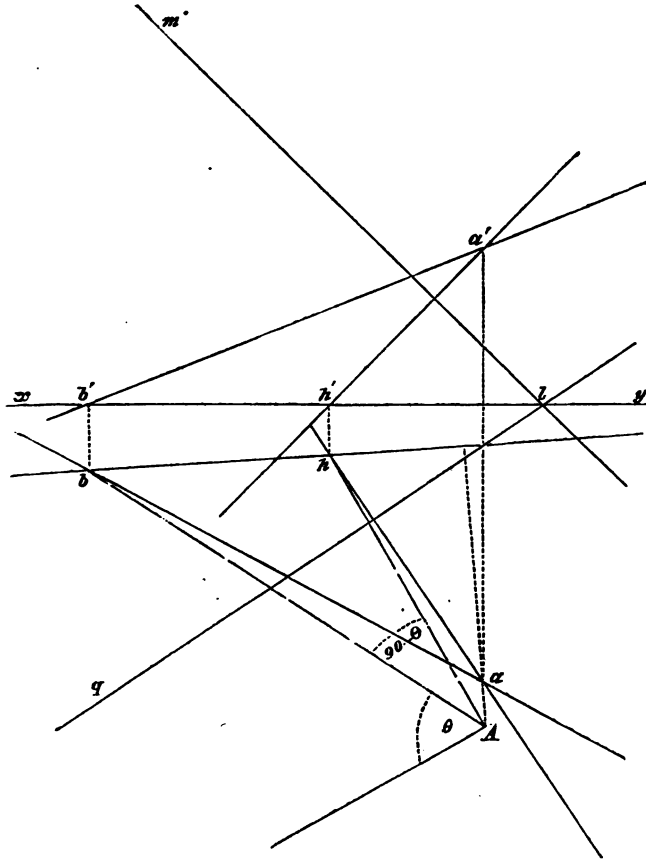
Determine this perpendicular ah, $a'h'$, by Problem XIII., and find its horizontal trace ha'; likewise bb' that of the given line; the line hb through these points is the horizontal trace of the plane of the lines. 'Rabat' their plane about hb bringing A into the horizontal plane. Ah, Ab, when drawn, give bAh the complementary angle or $90-\theta$, and θ is the angle sought.

Conversely:

To determine a straight line through a given point aa', *to make a given angle θ with a given plane* qlm'.

Through aa' draw ah, $a'h'$ at right angles to the given plane, and determine its horizontal trace hh'. Through

Fig. 24.



point h draw in any direction the line hb as the horizontal trace of the plane of the perpendicular and the required line. Fold this plane about hb into the horizontal one and through point A thus 'rabatted' let Ab making an angle

$90 - \theta$ with Ah meet the trace hb in point b. Then b' being found in xy and joined to a', the required line ab, $a'b$ is determined.

Observe. An infinite number of such lines ab, $a'b'$ can be determined to satisfy the given conditions; and their 'locus' is the right-conical surface having its circular base in the given plane and its generatrix inclined θ degrees to the same.

PROBLEM XXV.

To determine two planes which shall be perpendicular to each other and inclined at angles of θ and ϕ degrees respectively to the horizontal plane.

It will be obvious that the angle of inclination of the second plane cannot be less than the complement of that of the first; or ϕ not less than $90 - \theta$.

Therefore

if ϕ is greater than $90 - \theta$, there are two solutions:

...ϕ is equal to $90 - \theta$, there is one solution;

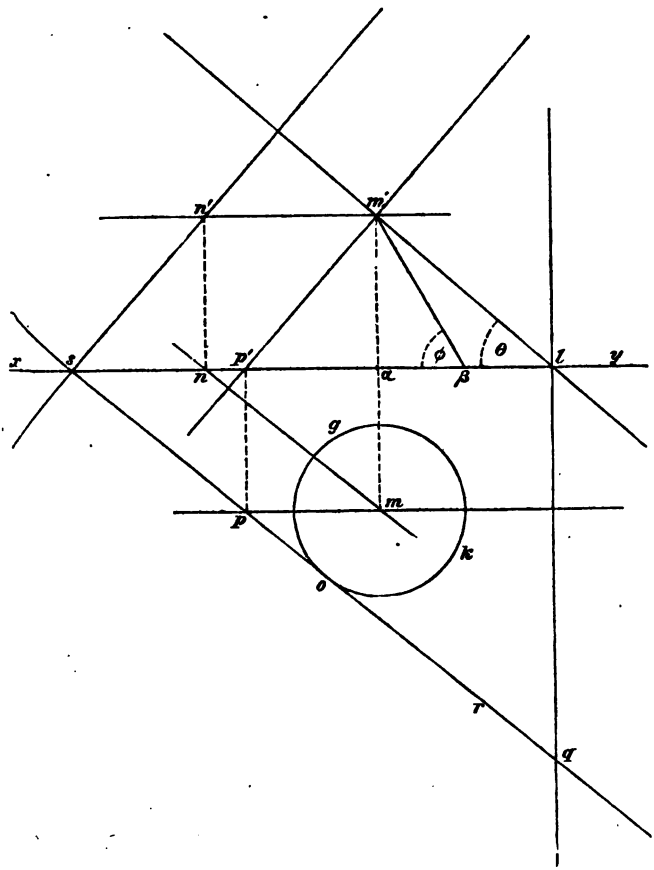
...ϕ is less than $90 - \theta$, there is no solution.

This will appear by what follows.

Draw the plane qlm' inclined θ degrees; also the straight line pm, $p'm'$ at right angles to it (Problem 13), and find the horizontal trace pp' of PM.

Then every plane which contains that line is at right angles to the plane θ (Eucl. XI. 18). Wherefore if such a plane revolve about PM, as an axis, until it makes an angle ϕ with the horizontal plane, it is the plane required.

Fig. 25.



Let it therefore be made to touch a right cone, vertex in any point mm' in PM and generatrix $m'\beta$ making an angle of ϕ degrees with its base gko. The horizontal trace of this plane must touch gko (Theorem XI.).

And since the plane is to contain PM, the horizontal trace rp of the plane must pass through p (Theorem X.).

Through the point mm' draw the horizontal line mn, $m'n'$ parallel to the trace rp, and determine its vertical trace nn'.

Through n' and s draw the vertical trace of the plane.

Observe that the plane rsn' by containing PM is perpendicular to plane θ; and, by touching the cone described, has the required inclination ϕ; it thus satisfies the conditions of the problem.

Note also that the inclination of PM is $90-\theta$; since if a straight line and an inclined plane be perpendicular to each other their inclinations are complementary.

COR. *To determine three planes each perpendicular to the other two.*

For the first and second, determine planes qlm' and rsn' as in the above problem.

Find their intersection mq, $m'q'$ by Problem 12.

The plane which is perpendicular to MQ is perpendicular to the planes which meet in this line (Eucl. XI. 19). Therefore such plane, determined by Problem 13, is the third plane required.

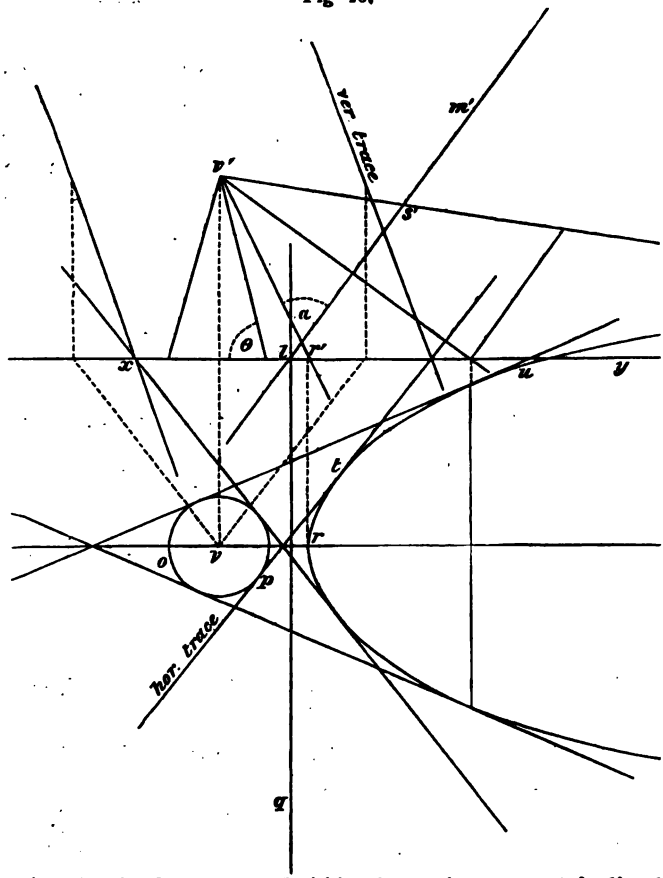
PROBLEM XXVI.

To determine a plane inclined at an angle of θ degrees and making a given angle α with a given inclined plane.

Let qlm' be the given inclined plane.

To fulfil both conditions, the required plane must touch two right cones having a common vertex in any point vv'

Fig 26.



with circular bases op and $b's'$ in the horizontal and inclined planes respectively*; the generatrix of the former making an

* b' is the point of intersection of the lines $v'r'$ and lm', and is not lettered in the fig.; also the plan of the circular base $b's'$ is not shown in the fig.

angle θ with its base, and that of the latter an angle of α degrees with the inclined plane.

The horizontal trace *pt* of the common tangent plane to the cones is (Theorem XII.) a tangent to the horizontal traces of the two conic surfaces, *i.e.* the trace *pt* touches the circle *op* and the ellipse *rtu*. Since four such lines can be drawn, as appears by the figure, it is obvious that in the case here indicated four planes can be determined which satisfy the given conditions of the problem.

Should the circle *op* fall within the ellipse, without touching it, the given conditions are impossible.

PROBLEM XXVII.

Given three lines ag, ak, ar *meeting in a point* a*; as the plans of the straight lines* AG, AK, AR, *each of which is at right angles to the plane of the other two; to determine an elevation of the lines.*

Since the plane of lines *AK*, *AR* is perpendicular to the line *AG* (Eucl. XI. 4), the horizontal trace, *cd*, of the plane of *AK*, *AR* is perpendicular to *ag*, the plan of *AG* (Theorem IX.).

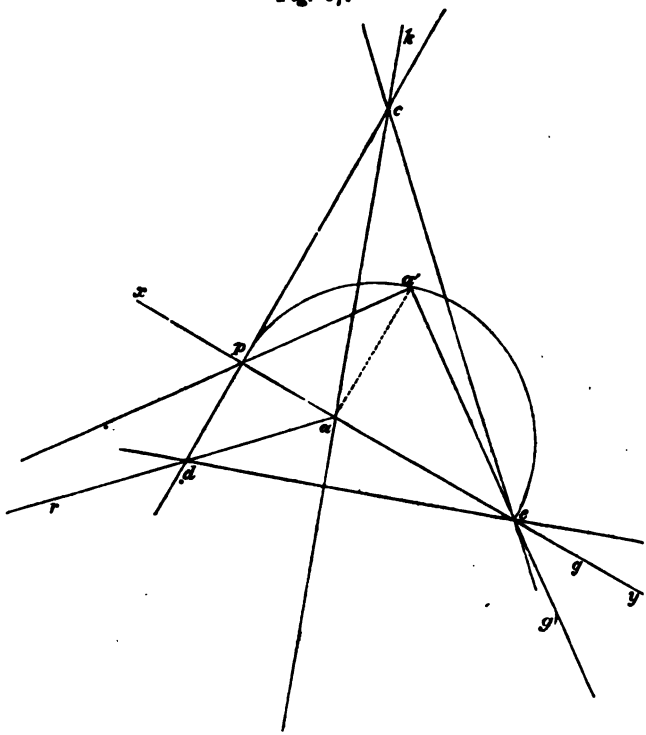
Therefore draw *cd* at right angles to *ga* produced, and meeting *ak*, *ar* in points *c* and *d*.

ce drawn through *c* at right angles to *ar*, and *de* drawn through *d* at right angles to *ak*, will meet *ag* in *e* and be the horizontal traces of the other two planes (Theorem IX.).

Take the projecting plane of the plan of *AG* for a plane of elevation, and let it revolve about the line *ag* as an *xy*.

This plane cuts the plane of *AK* and *AR* in *AP* or *a'p*, which is therefore perpendicular to *AG* or *a'g'* in the plane of elevation (Eucl. XI. def. 3).

Fig. 27.



Therefore on ep describe the semicircle $ea'p$, through a draw aa' perpendicular to xy, and ea', $a'p$ are the required elevations of the three lines (Theorem VI. 2).

PROBLEM XXVIII.

Given two points B *and* C *at unequal distances from a given plane, to determine a point* Q *in the plane at which the straight lines* QB *and* QC *shall make equal angles with the plane.*

From either of the points, as B, draw the straight line BR at right angles to the given plane (Problem 13), meeting it in S, determined by Problem 14.

Make $SR = SB$, point R being on the opposite side of the plane to B; join R, C, cutting the given plane in point Q: QB, QC are the required lines.

Exercises on Chapter III.

1. Determine the projections of four points, each 2 inches from the planes of projection, viz. one in each dihedral angle.

2. Determine four points each in a plane of projection and 3 inches from xy; no two of them to be on the same side of that line.

Supposing the loci or projecting lines of these points (Theorem IV.) to be 2 inches apart, determine the distance between any two points.

3. A point is 2·5 inches from each plane of projection, draw two lines through it inclined 50°; their plans making angles of 35° with xy. Problem 1, converse (2).

Determine the traces of the lines: also the angle which the lines make with each other. Problem 10.

4. Through a point 2 inches from each plane of projection draw lines inclined 30°, making angles of 40° with the vertical plane. Problem 3 and Theorem IX.

5. A straight line in the horizontal plane makes an angle of 45° with xy and is a trace of planes which are inclined 50°; find their vertical traces. Problem 3, "tangent planes to cones," Chap. VIII.

6. A regular hexagon, side 2 inches, has its diameter and adjacent side inclined 30° and 45° respectively: draw its plan and determine the inclination of its plane. Problem 20.

7. Draw two parallel planes 2 inches apart and inclined 60°. Problems 1 and 13, and Problem 25, *note.*

8. A plane is inclined 40°, and is at right angles to another which is inclined 70°: draw the traces of the planes. Problem 25.

9. The horizontal traces of two planes, inclined 45° and 30° respectively, make an angle of 60°: determine a third plane at right angles to the given ones. Problems 12 and 13.

10. Two planes are inclined 60° and 70° respectively and make an angle of 50° with each other: determine their vertical traces on a plane not perpendicular to them. Problem 26.

11. A regular pentagon, side 2·5 inches, has three of its angular points respectively 1, 2, and 3 inches high. Draw its plan. Problem 21.

12. A square, side 3 inches, lies in a plane inclined 45°, one side of the figure makes an angle of 40° with the horizontal trace of its plane. Draw its plan. Problem 18.

13. A plane is inclined 80° and makes an angle of 60° with a plane which is inclined 50°. Draw the traces of the planes. Problem 26.

14. The two traces of a plane make an angle of 50° with each other and make equal angles with xy: determine these latter angles and the inclination of the plane.

15. Three straight lines are perpendicular to one another and two of them are inclined 25° and 40°: determine the inclination of the third line. Problems 20 and 13.

16. The horizontal and vertical traces of a plane make angles of 30° and 55° respectively with xy; a straight line parallel to xy is 2 inches from the vertical and 3 inches from the horizontal plane: find the intersection of the given line and plane. Problem 14.

17. Two planes contain a right angle; one of them is inclined 60° and their intersection is inclined 50°. Draw the traces of the planes and find the inclination of the second. Problems 8 and 13.

18. The hypotenuse of a right-angled triangle is 4 inches long, and horizontal; the plans of the sides make an angle of 115° with each other: determine the lengths and inclinations of the sides. Can more than one set of answers be given? Eucl. III. 33, and Problem 20.

19. A straight line inclined 40° lies in a plane inclined 60°; determine a plane containing the given line and perpendicular to the given plane. Problems 8 and 13.

20. Through a point 3 inches high draw three planes, inclinations 35°, 45°, and 60° respectively; forming a tri-

hedral angle at the given point. Show the real magnitude of the angle of each face of the solid angle. Problems 12 and 10.

21. The plans of two lines contain an angle of 110°. The lines themselves are at right angles, and one is inclined 27°. Find the inclination of the other. *Science Exam. Hon.* 1872.

(1) Draw plans of the lines and make an elevation of one, assumed as that inclined 27°, on a vertical plane taken parallel to it.

(2) Determine a plane perpendicular to this line and containing the other. The point in which the horizontal trace of this plane meets the plan of the second line is the horizontal trace of the line whose angle of inclination is required.

(3) Determine this angle by Prob. 1.

22. Two lines are inclined at 30° and 40°. They are 2 inches apart where they are nearest together, and this line of 2 inches is inclined 28°. Show plan and elevation. *Science Exam. Hon.* 1873.

(1) Determine a line inclined 28°, and mark off a segment of this line 2 inches in length. Prob. 1.

(2) Draw two parallel planes at right angles to line of 28° passing through the extremities of the segment of 2 inches.

(3) From the point where line of 28° meets one plane draw a line lying in that plane and inclined at 40°, and from the point where the former line meets the parallel plane draw another line lying in that plane and inclined 30°.

The two lines so drawn are those required.

Note. Four pairs of lines fulfilling the conditions are possible.

Another solution may be based upon Problems 20, 1 (converse), 13 (converse) and 8.

23. Given a plane by its traces, and two lines, not passing through the same point, by their projections. The lines to be neither in the given plane nor parallel to it. Find the projections of a line which shall meet the two given lines, be parallel to the given plane, and at a given distance from it.

Determine a plane parallel to the given plane at a distance from it equal to the given distance of the parallel line from the same plane. Problem 13, Cor. 2.

Find the points of intersection of the two given lines with this plane. Prob. 14. The line joining the two points thus found is the line required.

24. Two horizontal lines AB, AC, contain an angle of 56°, a plane inclined at 30° contains AB, another inclined at 65° contains AC. Draw two lines passing through A, each inclined at 20° and lying one in each plane. Determine the angle between these two lines. *Science Exam. Hon.* 1872. Problems 4, 8 and 10.

25. A line TV 3·3 inches long has its extremities T and V in the horizontal and vertical planes respectively, it is inclined to the horizontal plane at 40°, and has its trace T at ·75 inch from the ground line. Draw its plan and elevation. *Science Exam.* 1868.

26. An indefinite line is inclined to the horizontal plane at 40° and makes an angle of 30° with the vertical plane of projection, the distance between its traces being 3·5 inches. *Science Exam. Hon.* 1869.

27. AB is a line parallel to the horizontal plane, AC is a line parallel to the vertical plane. The angles bac between the plans and $b'a'c'$ between the elevations are 120°. Find the real angle between the lines. *Science Exam.* 1871.

28. Determine a line inclined at 33° lying in a plane inclined at 50°. This is the orthographic projection on that plane of a line which makes an angle of 40° with it. Determine the projections of this latter line on the co-ordinate planes and its inclination. *Science Exam. Hon.* 1871.

29. Find the traces of a plane which contains three assumed points—one in each plane of projection and the third in space.

30. Bisect the dihedral angles formed by the three planes in the exercise 20 above.

31. Find the centre of a sphere of given radius that shall touch three assumed planes.

IV.

SOLIDS WITH THE INCLINATIONS OF THE PLANE OF ONE FACE AND OF ONE EDGE OR LINE IN THAT FACE GIVEN.

BEFORE commencing this chapter the student should refer to Chapter III. Problems 18 and 19.

1. *Plan and elevation of a pyramid,* 4 *inches high, with square base of* 2·5 *inches side, when its base is inclined* 50°, *and no side horizontal.*

(1) Determine by its traces a plane 50° at right angles to the vertical plane; *i.e.* draw its horizontal trace at right angles to, and its vertical trace at 50° with, the ground line.

(2) 'Rabat,' that is, fold this plane on its horizontal trace into the horizontal plane, and draw the square as then in the horizontal plane; *i.e.* simply draw the square, taking care that no side is parallel to *xy*.

(3) Lift the plane back to the required position and determine required plan and elevation of the square base. Remember that this plane being at right angles to the vertical plane will have all points in it shown in this elevation *in* the vertical trace (Theorem VI. 2). The square base will therefore have its *elevation* wholly in the vertical trace.

To draw this elevation, take the distances from the corners of the square to the horizontal trace when the plane is folded down, and set them along the vertical trace from its junction with xy. This will give these points in *elevation*. Their *plans* will lie in the perpendiculars to xy from their *elevations* (Theorem IV.) and also in the perpendiculars to the horizontal trace from the points as 'rabatted' (Theorem XX.). Thus let the point of which a' is the elevation when folded down or 'rabatted' be marked A. A perpendicular from a' to xy contains the plan a, as also does the perpendicular from A to the horizontal trace. The plan a therefore lies in the intersection of these perpendiculars; in other words, these perpendiculars are loci of the point a, and their intersection gives the point itself.

(4) The axis will be drawn from the centre of the base in plan and elevation at right angles to the traces, and being parallel to the vertical plane will be set in true length *in elevation*. It only remains to join the proper points, and the projections of the solid are completed.

2. *Plan and elevation of a cube of* 2·5 *inches edge, when one face is inclined* 60°, *and no side horizontal.*

3. *Plan and elevation of a pyramid,* 4 *inches high, with regular hexagonal base of* 1·5 *in. side inclined* 70°, *no side horizontal, and its lowest corner* 1 *inch above the ground.*

4. *Plan and elevation of the pyramid in the last problem when the base is inclined* 70° *and one side of the base* 45°.

In this problem there is given not only the inclination of the plane of the base, but also of one edge in that base.

LIMITS. The problem is therefore to draw a line of given inclination in a plane of given inclination, and this is always possible if *the inclination of the line is not greater than that of the plane.*

(1) Determine by its traces the required plane inclined 70° and at right angles to the vertical plane. Assume any point A in the plane; its elevation a' is by the position of the plane in the vertical trace; its plan a on a perpendicular to xy from a', and for convenience it, *i.e.* the plan a, may be assumed at the point where this perpendicular meets xy; and from a' draw $a'm$ to meet the ground line at the given angle 45°. Rotate the triangle thus formed on the vertical line $a'a$, the projector of the point A, until the point m meets the horizontal trace. As A and m are now both in the given plane, the line joining a and m will be the plan of Am and will fulfil the conditions by being the plan of a line wholly in plane 70° and itself inclined 45°.

(2) 'Rabat' plane 70° on its horizontal trace into the horizontal plane. On the line Am, thus brought into the horizontal plane, describe the required hexagon, and complete plan and elevation of the whole as in former problems.

5. *Plan and elevation of a cube of* 2·5 *inches edge when the plane of one face is inclined* 65° *and one diagonal of that face* 25°.

6. *Plan and elevation of a tetrahedron of* 3 *inches edge with one face* 70° *and one edge in that face* 40°. *Determine also the inclinations of the other edges of the solid.*

7. *Plan and elevation of hexagonal right prism when the plane of one of its faces* ABCD *is inclined* 50° *and the edge* AB *of the base inclined* 35°.

(1) Determine the plan of rectangular face $ABCD$ as in preceding problems.

(2) It will next be necessary to determine the plane of the base, which, by the definition of the solid, is perpendicular to the plane of the face. The line AB already determined is in the base, being common to both planes, and its horizontal trace will therefore be a point in the horizontal trace of the required plane of base. And as the projections of a perpendicular to a plane are perpendicular to its traces, the horizontal trace of the plane of the base may be at once drawn through this point at right angles to the long edges of the prism, that is, in this case to *bc* or *ad*.

(3) Take a ground line at right angles to the horizontal trace of plane of base, and set up the height of A or B (both known points) in the new elevation. By the position of the new plane of elevation, the vertical trace will contain the elevations of both A and B, and therefore can at once be drawn.

Fold the plane of the base about its horizontal trace to bring AB down into the horizontal plane. On the AB thus rabatted describe hexagon and complete as in former problems.

8. *The plane containing one edge of a tetrahedron and bisecting another is inclined at* 50°, *and the former edge is inclined at* 30°. *Draw plan and elevation of the solid, the edge being* 3 *inches.* Science Exam. Hon. 1870.

(1) Determine plane 50° containing line inclined 30°, and 'rabat' it.

(2) Measure off on the rabatted line 3 inches, and on this segment as a base draw an isosceles triangle having its

equal sides equal to the altitude of an equilateral triangle of 3 inches side. The figure thus drawn is the true form of the section of the tetrahedron made by the given plane.

(3) Rerabat plane, and from vertex of triangle draw a perpendicular to the plane and produce it on both sides to equal distances of 1·5 in. On joining the extremities of this perpendicular with the extremities of the line of 3 inches measured off on line of 30°, we obtain the projections of tetrahedron required.

9. *Plan and elevation of octahedron* 3 *in. edge.*

α. *Plane of one face inclined at* 65° *and one edge of that face at* 30°.

β. *Plane containing two axes inclined at* 70° *and one edge of the solid in that plane at* 35°.

γ. *When two of its axes are inclined at* 60° *and* 20° *respectively.*

10. *Pentagonal pyramid, side of base* 1·5 *in., axis* 3·5 *in., when the plane containing the axis and one of the slant edges is inclined at* 50° *and the axis at* 25°.

11. *Octagonal right prism, side of base* ·75 *in., axis* 3 *in., when the latter is inclined at* 30° *and one side of the base at* 27°.

V.

SOLIDS WITH THE INCLINATIONS OF TWO ADJACENT EDGES OR LINES GIVEN.

LIMITS. It must be remembered that *the sum of these inclinations with the angle contained by the two edges must not exceed* 180°.

For these problems refer to Chapter III., Problem 20.

1. *Plan and elevation of a cube when the inclinations of two adjacent edges as* BA, BC, 20° *and* 48°, *are given.*

(1) Draw the square $ABCD$ to represent the face rabatted into the horizontal plane. Then to find a horizontal in the plane of this face:—At any point E in BA, or BA produced, make the angle BEF 20°. Then if a perpendicular be drawn from B to EF, it will be the length of the line expressing the height to which B, when in required position, will be raised above horizontal plane containing point E. From B as a centre with this perpendicular for radius, describe a circle. The tangent to this circle to make an angle of 48° (*i.e.* the inclination of the other edge) with BC will give a point H where it meets BC in the *horizontal* required. E being another point in it, the straight line through H and E will be the horizontal sought in plane of face $ABCD$.

(2) Assume a plane of elevation at right angles to this

inclined plane of face *ABCD*. The height to which point *B* is to be raised being known, a line of level that distance above the *xy* taken will contain the elevation of *B*. The rest of the work is similar to that in preceding problems.

NOTE. To understand the principles of this construction, the student had better cut a card-board model, by which to illustrate the folding up and down of the plane of the angles and of the lines.

2. *Plan and elevation of a tetrahedron when two of the adjacent edges are inclined* 30° *and* 45°.

3. *Plan and elevation of hexagonal right prism when the two edges* AB, AD, *in one face are inclined respectively* 45° *and* 27°, AB *being the edge of the base.*

Determine the plan of the rectangular face *ABCD* as in Problem 1, and refer to explanations of Problem 7, Chapter IV., to find the plane of base and to complete the projections of the solid.

4. *Octagonal prism,* 3 *in. long, side of base* 1 *in., when its axis is inclined at* 40° *and an edge of the base at* 27°.

5. *Cube,* 2·5 *in. edge, when two diagonals of the solid are inclined at* 40° *and* 35° *respectively.*

6. *Octahedron* 3 *in. edge.*

α. *When two adjacent edges in one face are inclined at* 32° *and* 40°.

β. *When two adjacent edges not in one face are at* 20° *and* 48°.

7. *A regular pentagonal pyramid,* 2·5 *in. side, axis* 3·5 *in., has three of the angular points of its base taken in order,* 1 *in.,* 2 *in., and* 3 *in. high respectively. Plan and elevation.*

VI.

SOLIDS WITH THE INCLINATIONS OF TWO ADJACENT FACES GIVEN.

FOR these problems refer to Chapter III., Problem 25.

LIMITS. Case 1. If the given planes are at right angles to each other, their inclination being θ° and ϕ°, $\theta^\circ + \phi^\circ$ must not be less than 90°.

Case 2. If the given planes are not at right angles to each other, refer to Chapter III., Problem 26.

1. *Plan and elevation of a cube of* 3 *inches edge, when the inclinations,* 45° *and* 70°, *of the planes of two adjacent faces are given.*

(1) Determine the traces of plane 45° assumed at right angles to the vertical plane.

(2) Determine the projections of a perpendicular to plane 45°, in order to determine plane 70°, since every plane containing this perpendicular is at right angles to the former plane.

(3) Determine *horizontal trace* of plane 70°. To do this, make any point in the perpendicular—*e.g.* that in which it pierces the plane of 45°—the vertex of a right cone, generatrix inclined 70°, base in horizontal plane. Then, since

every plane touching this cone is inclined 70°, a tangent to the circular base of the cone will be the horizontal trace of such a plane, and if it be drawn as well through the horizontal trace of the perpendicular to plane 45°, it will be the horizontal trace required, *i.e.* of a plane perpendicular to plane 45° and inclined itself 70°.

(4) Determine the *plan* of the intersection of the two planes. One point in this plan will be at the intersection of the horizontal traces of the two planes, and a second point will be the plan of the point in which the perpendicular pierces the plane. The line drawn through these two points will be the plan of the intersection required. Rabat one of the planes (say that at 45°) containing this line into the horizontal plane, draw the square face of the cube on the line so rabatted, and determine plan of the face therefrom. Complete plan and elevation of the cube as in former problems.

2. *A prism* 3 *inches long has an equilateral triangle of* 2·3 *inches side for its base; draw its plan and an elevation of it when the planes of its base and one face are equally inclined at* 60° *to the horizontal plane.* Science Exam. Hon. 1869.

3. *Plan and elevation of a prism with base a regular hexagon. Plane of one face inclined* 48°. *Base* 68°.

(1) Determine the plan of the rectangular face of prism as for the face of the cube first determined in problem 1.

(2) On a vertical plane at right angles to plane 68°, *i.e.* with its ground line taken at right angles to the horizontal trace of plane 68°, determine the vertical trace of plane 68°, and rabat this plane into the horizontal plane. On *AB*, the edge common to plane of face and plane of base so rabatted, describe the regular hexagon.

Determine plan therefrom and complete problem as in former cases.

4. *Plan and elevation of a tetrahedron. Planes of adjacent faces* θ *and* ϕ.

For example, let $\theta = 45°$ and $\phi = 75°$.

(1) Determine the angle α between the two adjacent faces of the solid.

(2) Draw the traces of plane θ, taking the plane at right angles to the vertical plane of elevation.

(3) Determine a second plane inclined ϕ and making an angle of α with plane θ, Chapter III. Problem 26. The intersection of these two planes θ and ϕ is the indefinite edge of the solid. Rabat plane θ or ϕ, and on the intersection so rabatted mark off the length of the given edge of the tetrahedron, describe the face upon it, and proceed as in former problems.

5. *Plan and elevation of a tetrahedron with one face vertical and an adjacent face inclined* 50°.

Determine the angle θ between adjacent faces.

Determine a plane inclined 50° to the horizontal plane and θ to the vertical plane by Chapter III. Problem 7.

On the vertical trace place $a'b'$ the real length of an edge of the tetrahedron, and on it describe the equilateral triangle $a'b'c'$. Find the centre v'. $a'b'c'v'$ is the required elevation.

In plan the vertical face $a'b'c'$ will be a straight line abc parallel to xy, and the plan v of the vertex will be a point in the projector drawn from v' perpendicular to xy at a distance in front of abc equal in length to the axis of the tetrahedron. Lines joining v with the points abc of the vertical face complete the plan of the solid.

NOTE. This problem should also be worked by the general method as given in Problem 4.

6. *Substitute the octahedron for the tetrahedron in Probs.* 4 *and* 5.

7. *Pentagonal pyramid in Prob.* 10, *Chap.* IV.

α. *When the planes of two adjacent faces are inclined at* θ^0 *and* ϕ^0.

β. *When the plane of the base is inclined at* θ^0 *and of one face at* ϕ^0.

8. *Plan and elevation of right prism, regular hexagonal base, when the plane of one face is inclined at* 50^0 *and its axis at* 35^0. *To be worked by the method given for this Chapter.*

VII.

PROPERTIES OF CURVED SURFACES AND PROBLEMS BASED THEREON.

THE investigation of the properties of curved surfaces can be greatly simplified by making certain assumptions as to the mode in which these surfaces have been generated by the motion of a line, straight or curved, of constant or variable magnitude, according to a definite law.

The line whose motion generates the surface is called the *generatrix.* Any lines which limit or direct the motion of the generatrix are called *directrices.*

All surfaces having straight lines for their generatrices are called *Ruled Surfaces.*

Ruled Surfaces are divisible into two classes; *Developable Surfaces* and *Twisted* or *Skew Surfaces.*

Any two *consecutive generatrices* of a Developable Surface lie in one plane. This is not true of Twisted or Skew Surfaces. The former can therefore be laid flat, or developed, without extension or rupture of any part, by turning each element of the surface successively, about the consecutive generatrices, into the plane of development. The latter are not developable.

The simplest of all Ruled Surfaces is the *plane*, which is generated by a straight line moving parallel to itself along another straight line fixed in position.

To understand clearly what is meant by a Developable Surface, observe that since the consecutive generatrices lie in one plane they must either meet or be parallel. Let A, B, C, D, &c. be a series of consecutive generatrices of a Ruled Surface, and let B meet A in the point 1, C meet B in 2, D meet C in 3 and so on. The limit of the points 1, 2, 3,...n, will be a curve, generally of double curvature, called the *Edge of Regression*, to which every generatrix of the surface is a tangent. The parts of the surface generated by the tangent line to the edge of regression on each side of the point of contact, form what are called the *two sheets* of the surface.

In the *cone* the points 1, 2, 3, &c. coincide, and the edge of regression becomes, therefore, a point—*i.e.* the vertex of the cone.

In the *cylinder*, the generatrices being parallel, the edge of regression is at an infinite distance.

From the mode of generation, it is evident that through every point on a Ruled Surface one straight line, at least, can be drawn coinciding throughout its whole length with the surface.

Some surfaces may be conceived as generated in a variety of ways, but there is one class which it is most convenient to regard as generated by the motion of a line about a fixed axis in such a manner that every point in the generating line is always at the same distance from two fixed points in the axis. These are called *Surfaces of Revolution*. The generating line may be straight, or of

single or double curvature. The treatment of these surfaces is generally facilitated by assuming the generatrix to be the line in which a plane containing the axis would cut the surface. Such a section is called a *meridian* section.

The limits of this book do not admit of an exhaustive treatment of the properties of curved surfaces, but the student who wishes to pursue the subject, by the aid of Descriptive Geometry, into its higher developments, is strongly advised to do so on the lines indicated in the foregoing general remarks.

PROBLEMS ON THE PROJECTION OF CURVED SURFACES.

The construction of special generatrices of curved surfaces is always a problem of importance, and, generally, the projection of a curved surface is determined by the projection of a sufficient number of its generatrices.

THE SPHERE.

DEF. A sphere is the surface generated by the revolution of a semicircle about its diameter, which remains fixed in position during the motion.

A sphere may, therefore, be regarded as the locus of all points in space which are equidistant from a fixed point.

THEOREM I. Every orthographic projection of a sphere is a circle equal in radius to the radius of the sphere.

THEOREM II. Every plane section of a sphere is a *circle*. That cut by a plane passing through the centre is a circle equal in radius to the radius of the sphere, hence called a "*great circle*." Other plane sections are *small circles*.

1. *To determine the centre and radius of a sphere which shall*

pass through four given points, A, B, C, D, *not in the same plane, and no three whatever of which are in the same straight line.*

THEOREM. One sphere and one only is possible.

FIRST METHOD. If the line joining any two of the points be bisected by a plane perpendicular to it, since every point in that plane is equidistant from the extremities of the line which it bisects, the plane must contain the centre of the sphere (Theorem XVIII. page 11). For the same reason if two other planes be assumed similarly bisecting any other two of the lines joining the given points, three planes will have been determined, each containing the required centre. This will therefore be the point which is the common intersection of the three planes.

SECOND METHOD. Determine the centre P of the circle which passes through any three of the points, as A, B, C, and through the point P draw a perpendicular to the plane of these three points. This perpendicular will contain the centre of the sphere. Similarly, determine the centre Q of the circle which shall pass through the fourth point D and two of the others, as A, B. The perpendicular through Q to the plane of A, B, D, will also contain the centre of the sphere. The point O in which the perpendiculars meet will therefore be the centre of the sphere required.

The radius of the sphere is the true length of a line from O to any one of the given points. Prob. 1, Chap. III.

NOTE. To simplify the problem, three of the points may be taken in one of the planes of projection.

2. *Three spheres,* A, B, *and* C, *of* 1·6 *in.*, ·8 *in.*, *and* ·4 *in. radii respectively, lie on the horizontal plane, each touching the other two. Plan and elevation.*

Begin with an elevation of the spheres *A* and *B* on a vertical plane assumed parallel to a vertical plane containing their centres. In this position the elevations will be two circles resting upon *xy* and touching each other. Draw two similar elevations of *C*, one touching *A* and the other touching *B*. Take the horizontal distances between the elevations of the centres of *A* and *C*, and of *B* and *C*, and from the plans of the centres of *A* and *B* with these distances for radii respectively, describe arcs. The intersection of these arcs will determine the plan of the centre of *C*, the third sphere. Its elevation can then easily be determined.

3. *To determine the projections*, pp', *of a point*, P, *on the surface of a given sphere.*

Let *oo'* be the projections of the centre *O* of the sphere, and *p* the plan of the given point. The plan *p* must of course be assumed somewhere within the circle which is the plan of the sphere. The elevation *p'* of *P* will be in the projecting line drawn from *p* at right angles to *xy*, and its height may be found by taking an auxiliary vertical section plane through the point *P* and the centre *O*, and rabatting it into one of the planes of projection.

4. *A sphere of* 1 *in. radius revolves uniformly about its fixed vertical diameter. A tracing point is carried uniformly over its surface in a fixed vertical plane from the highest to the lowest point of the sphere, during the time in which the latter makes one complete revolution. Show the projections of the curve. Centre of sphere* 2 *in. high.*

(1) Beginning at the highest point, divide one half of a vertical great circle of the sphere into any number of equal parts. Through these points take a series of auxiliary

horizontal planes cutting the sphere. The sections in plan will be circles, and in *elevation*, straight lines parallel to xy.

(2) Divide the *complete* circle, that is the plan of the sphere, into the same number of equal parts as the half of the great circle above is divided into, and draw the radii of the circle from each of these points. Starting from the highest point of the surface, construct the plan of the curve through the points where the first circle is cut by a radius, the second circle by the next radius, and so on to the circumference of the plan, and then back again to the lowest point.

(3) For the elevation, project the points through which the plan of the curve is drawn to the elevations of the horizontal sections in which they lie.

The case should also be worked in which the velocity of the revolving sphere is doubled and the *vertical* velocity of the tracing point constant.

5. *The plane of a circle* 2·5 *in. diameter is inclined at* 25°. *Its centre is* 2·8 *in. above the horizontal plane. Determine the sphere which resting on the horizontal plane has this circle on its surface.* Science Exam. Hon. 1873.

(1) Determine plane of 25° and the projections of the circle lying therein.

A straight line drawn from the centre of the given circle at right angles to its plane will pass through the centre of the required sphere.

(2) Draw projections of a line at right angles to given plane from centre of given circle and find the horizontal trace of this line.

(3) Draw a vertical plane containing the line determined in (2). This plane will be perpendicular to plane 25°.

and its horizontal trace will pass through the horizontal trace of the perpendicular to this plane (2) and will also be at right angles to the horizontal trace of the given plane.

The vertical plane will cut the given circle at two points which are the extremities of a diameter, shown in plan as the minor axis of the ellipse which is the plan of the circle. These points are, moreover, two points in a great circle of the sphere which has the given circle on its circumference and which rests on the horizontal plane at a point in that great circle.

The plane of this great circle is vertical and coincides therefore with the vertical plane just drawn. The horizontal trace of this plane is a tangent at the point of the great circle mentioned above as that in which the sphere touches the horizontal plane.

(4) Rabat this vertical plane and with it the two points into the H.P.

(5) Determine a circle passing through these points and touching the horizontal trace of the vertical plane (3). The centre of this circle is that of the sphere constructed, and its radius that of the sphere required.

Find plan and elevation of the centre and describe circles therefrom with the radius found.

The circles so drawn are the required projections of the sphere.

6. *Determine three points*, A, B, C, *at unequal distances from an assumed oblique plane, and construct the sphere which shall touch the oblique plane and pass through the three points.*

NOTE. The three points must be on the same side of the plane and must not lie in the same straight line.

(1) Find the projections α, β, γ, of the three points A, B, C, upon the oblique plane.

(2) Determine a plane passing through A, B, C, and find its intersection with the same plane.

(3) Rabat the oblique plane with its contained points, α, β, γ, and the intersection found in (2), into the horizontal plane.

Then, since the lengths of the projectors, αA, βB, γC, are known, the points A, B, C, relatively to the rabatted oblique plane, can be set up, the inclination of their plane determined, and a circle described through them.

(4) Find the centre and radius of the sphere that touches the rabatted oblique plane and has this circle on its surface, as in the preceding problem, and lift the oblique plane, and the centre of the sphere, back into position.

For another construction see Bradley's *Elements of Geometrical Drawing*, Part II. Plate XXXII.

7. *Three spheres*, A, B, C, *of* 2·3, 1·9, *and* 1·3 *inches diameter, rest upon the horizontal plane, each touching the other two. Determine plan and elevation of a fourth sphere,* ·75 *inch radius, touching all three.*

(1) Draw plan and elevation of the three spheres, A, B, C, as in Problem 2 above.

(2) Determine an elevation of two of the spheres, as A, B, on a vertical plane parallel to the line joining their centres. The elevation will be two circles touching at a point on their circumferences which is a projection of the point in which the spheres themselves touch.

(3) Draw a third circle of ·75 inch radius touching each of the two circles determined above. Consider the

circle last drawn as a projection of the sphere of ·75 inch radius which is required to touch all three given spheres. From the drawing it will be apparent that the elevation drawn is that of a sphere ·75 inch radius touching two only of these spheres, viz. *A* and *B*. Conceive the sphere of ·75 inch radius to roll round the two spheres *A*, *B*, and preserve its contact with each during the revolution. The centre of the revolving sphere will describe a circle in a plane at right angles to the line *AB*, which circle, in elevation, will be a straight line drawn from the centre of the circle of ·75 inch radius perpendicular to *a'b'* and produced to an equal distance on the opposite side of this line.

NOTE. The circle described by the centre of the revolving sphere is the locus of the centres of all spheres of ·75 inch radii which touch each of the spheres *A*, *B*.

(4) Determine plan of this circle. The plan will be an ellipse with its major axis at right angles to *ab*.

(5) Proceed similarly with one of the two spheres taken above and the third, *C*, and determine the projections of the locus of the centre of a sphere of ·75 inch radius in simultaneous rolling contact with both. The plan of this locus will likewise be an ellipse, and the two points where the loci intersect will be shown in plan by the intersection of the ellipses.

The points thus determined are the centres of spheres of ·75 inch radii which satisfy the required conditions.

For practice another example may be taken in which the three spheres, *A*, *B*, *C*, are at different heights above the horizontal plane and not in contact, taking care not to remove the centres so far apart as to render the problem impossible.

NOTE. After working Chapter x. the student will see that these points are also the two intersections common to three spheres described from the centres of the three given spheres A, B, C, with radii equal to that of $A + \cdot 75$, $B + \cdot 75$, $C + \cdot 75$, from the centres of A, B, C, respectively.

8. *Determine two points on the surface of a given sphere, and find the projections of the shortest line that can be traced on the sphere between them.*

NOTE. It will be a segment of the great circle passing through the points.

THE CYLINDER.

DEF. A cylinder is the surface generated by a straight line which moves parallel to itself and always passes through a given curved line.

A right cylindrical surface is one whose plane section at right angles to its generatrices is a circle.

The traces of a cylinder are the curves in which it intersects the horizontal and vertical planes of projection.

The directrix of a cylindrical surface may be a line of single or double curvature, but it can in every case be reduced to a plane curve—its *trace*, for example. Generally, any curved line whatever can be regarded as the intersection of two cylindrical surfaces whose generatrices are respectively perpendicular to the horizontal and vertical planes of projection, and directrices plane curves lying in those planes. Conversely, the projection of any curved line whatever on a plane is the trace, on that plane, of a cylindrical surface whose generatrices are perpendicular to the plane of projection, and directrix the given curved line.

A cylinder is given when one *trace* and a *generatrix* are given.

1. *Given* ab, *the major axis of the ellipse forming the section of a right cylindrical surface by the horizontal plane when the axis of the surface is inclined* $\theta°$; *determine the base and the plan of the frustum.*

NOTE. The minor axis of the ellipse is equal to the diameter of the required base.

The vertical plane through *ab* contains the axis of the cylinder and is at right angles to its base. Rabat this plane into the horizontal. Bisect *ab* in *o*, and draw *op'* making the angle *bop'* equal to $\theta°$. Through *p'* draw *l'p'm'* at right angles to *op'*, and meeting parallels to that line through *a* and *b* in *l'* and *m'*. Then *l'm'* gives the length of the required minor axis *cd*. From these data the curve may be readily drawn.

Through *p'* a perpendicular to *ab* produced meets that line in *p*, which is the centre of the plan of the base of the cylinder. The plan of this base is likewise an ellipse, having its major axis *nq* equal and parallel to the minor of the former, its own minor *lm* being determined by perpendiculars on *ab* drawn from *l'* and *m'*. Tangents to these curves parallel to the plan of the axis complete the plan of the frustum of the cylinder.

2. *Given an ellipse as the horizontal trace of a right cylindrical surface. Find its plan, and an elevation on a vertical plane making an angle of* 30° *with the plan of the axis.*

(1) Describe a semicircle on the major axis *ab*, and from one extremity, as *a*, set off the length of the minor

axis, *cd*, as a chord of the semicircle to *m*. The line drawn from *o*, the centre of *ab*, at right angles to *am*, is an auxiliary elevation of the axis of the required cylinder on a vertical plane passing through *ab*.

Parallels to the elevation of this axis from *a* and *b* complete the auxiliary elevation of the indefinite cylinder. Tangents to the ellipse, parallel to its major axis, complete the indefinite plan.

(2) Draw *xy* making an angle of 30° with the plan of the axis, for the new elevation. Assume any point *P* in the auxiliary elevation of the axis, and find its plan *p*. From plan *p* draw a projecting line at right angles to *xy*, and measure off on this line above *xy* the height of *P*,—the point thus found will be the elevation, *p'*, on the new vertical plane, of the point *P* in the axis. A line from *p'* to the point *o'* where the perpendicular from *o* meets *xy*, will be the elevation of the axis on the new vertical plane.

The elevation of the cylinder may be now completed by drawing two lines parallel to the elevation of the axis, one on each side, at distances equal to the semi-minor axis of the ellipse.

3. *To determine the projections*, pp', *of a point*, P, *on the surface of a cylinder.*

Let *p* be the plan of a given point, *P*, on the surface of the given cylinder. Through *p* draw the plan of a generatrix of the surface, and find its elevation. The point *p'*, in which the projecting line from the plan *p* cuts the elevation of the generatrix, will be the elevation of *P*.

4. ***Draw the cylinder which would envelope a sphere of 1·5 in. radius and centre 2·5 in. above the ground. Axis of cylinder inclined 50°, and making in plan an angle of 30° with xy. Show the circle of contact and the horizontal trace of the cylinder.***

(1) Draw projections of the sphere and plan of the axis of the cylinder according to the conditions.

(2) Use the plan of the axis as the ground line for an auxiliary V.P., in which draw a second elevation of the sphere. A line drawn through the centre of this elevation of the sphere, to meet the new ground line at the given inclination, will be the elevation of the axis of the required cylinder, and parallels to it, tangential to the circle which represents the sphere, will determine the major axis of the ellipse in which the cylinder cuts the horizontal plane. The minor axis of the curve will be marked off by parallels to the axis of the cylinder in plan, tangential to the circle which is the plan of the sphere. From the new or auxiliary elevation, an elevation on the original vertical plane can now be easily determined.

(3) The circle of contact is shown in the auxiliary elevation as a straight line, and from this can readily be determined in plan, and from thence on the original vertical plane.

NOTE 1. The horizontal trace of the cylinder might also be found by means of generatrices drawn through a number of assumed points in the circle of contact.

NOTE 2. The trace of this cylinder is the boundary of the shadow of the sphere, cast upon the horizontal plane by rays of light parallel to the generatrices of the

cylinder; and the "circle of contact" marks the boundary of light and shade on the sphere. See *Shadows*, Chap. IX.

5. *The axes of two equal cylindrical surfaces are inclined in opposite directions at* 25°, 50°, *the line perpendicular to both these axes is* 3 *inches long and inclined at* 30°, *the two surfaces touch each other in one point only. Draw plan of cylinders showing point of contact. Elevation at pleasure.*

Science Exam. Hon. 1869.

(1) Determine projections of the axes and their common perpendicular by Problem 22, Exercises on Chap. III.

(2) Bisect the plan and elevation of the line of 3 inches which is perpendicular to the axes. The points of bisection are the projections of the point in which the cylinders touch.

(3) The plans of the cylinders can be completed by making auxiliary elevations of the axes on vertical planes assumed containing these axes. Lines parallel to the elevations of the axes and in the same vertical planes with them, at distances of 1·5 in. on each side, meet the horizontal plane in points, on the plans of the axes produced, which are the extremities of the major axes of the elliptic traces of the cylinders on the horizontal plane. The semi-minor axes of the same sections are equal to 1·5 in., the radius of the cylinders, and hence the ellipses can be drawn and the plans and elevations completed.

6. *The centre,* O, *of a sphere of* 1 *in. radius is* 1·5 *in. above the horizontal plane and* 2·5 *in. in front of the vertical plane. A straight line in the horizontal plane* 2·25 *in. from the plan,* o, *of the centre of the sphere and making an angle of* 30° *with* xy, *is the plan of the indefinite axis, inclined at* 50°,

of a right cylindrical surface touching the sphere. Determine the traces of the surface and the point of contact with the sphere, when the latter is without the cylinder.

(1) Determine a plane containing the centre O of the sphere and the given axis. This plane cuts the sphere in a great circle, and the touching cylinder in a generatrix which is a tangent to this circle at a distance from the axis equal to the semi-diameter of the cylinder required.

(2) Rabat this plane, with the great circle and given axis, into the horizontal plane. A line touching the rabatted great circle parallel to the rabatted axis will give the point P of contact, from which the projections, pp', may be readily obtained. The distance between these parallels will give the semi-diameter of the cylinder, whence the traces and the projections of the surface may be determined.

NOTE. The student will understand the methods by which touching surfaces are determined better after he has worked systematically through the problems on tangent-planes to curved surfaces given later on.

Two surfaces are said to touch at a point when they have a common tangent plane at that point. A given straight line may, in general, be the axis of two right cylinders touching a given sphere. When the given axis passes through the centre of the sphere one cylinder only is possible, and it in this case *envelopes* the former surface. When the axis is a tangent to the sphere one cylinder only is possible, and it contains the sphere. If the axis be entirely free from the sphere two cylinders—one including and the other excluding the sphere—can be drawn. When the axis meets the sphere in two points and does not pass through its centre, two tangent cylinders—one containing and one intersecting the sphere—are determinable.

7. *A right cylindrical surface of 2 in. diameter revolves uniformly about its axis fixed in position, and a line is traced upon it by a point which moves over the surface parallel to its axis at a uniform velocity. Draw the projections of the curve formed when the tracing point moves 1 in. forward during the time of each complete revolution of the cylinder.*

Work two cases, (1) *when the axis is vertical,* (2) *when the axis is oblique, and a development.*

NOTE. This curve is the well-known *helix* or *screw-thread.* One of its properties is that the tangent line to the curve at any point is inclined to the generatrix of the surface through that point at an angle constant for every part of the curve. Hence when the cylinder is developed the curve becomes a straight line inclined at a determinable angle, α°, to the line which is the developed circular base of the cylinder. This angle is called the *angle of the screw,* and is equal to that which the tangent line at any part of the curve makes with the plane of the base of the cylinder. The distance, P, through which the tracing point moves while the cylinder makes one revolution, is called the *pitch* of the helix. Obviously $P = 2\pi r \tan \alpha$.

To work the problem, divide a circular base of the cylinder into any number of equal parts and draw generatrices through the points of division. Let N be the number of parts into which the circular base is divided. Starting from one of the points of division, measure off from the base, on the successive generatrices, distances successively equal to $\frac{P}{N}$, $2\frac{P}{N}$, $3\frac{P}{N}$.....$N\frac{P}{N}$. P in this case being 1 in.

Join the projections of these points for the projections of the helical curve.

The development presents no difficulty.

8. *Assume any two points on the surface of the cylinder in the last problem not in the same generatrix, and show the projections of the shortest line that can be traced on its surface between them.*

Develop the surface with the two contained points, and join them by a straight line. The points in the development in which this line meets certain of the generatrices can then be easily transferred to the corresponding projections of these generatrices, and the points so found be joined for the projections of the required line, which, in this case, is clearly a portion of a helix.

NOTE. This problem may be taken as a type of the method by which the shortest line between any two points on any developable surface whatever may be found, or any given line laid upon the surface. See Prob. 7 on "The Cone."

THE CONE.

DEF. A cone is the surface generated by the motion of a straight line which always passes through a fixed point and a given curved line.

The fixed point is called the *vertex* of the cone.

The directrix may be of single or double curvature, but every conic surface may be represented by a plane directrix, a fixed point, and a rectilineal generatrix.

If the generatrices extend beyond the vertex, a second similar cone is generated. These simultaneously generated cones are called the *two sheets* of the surface.

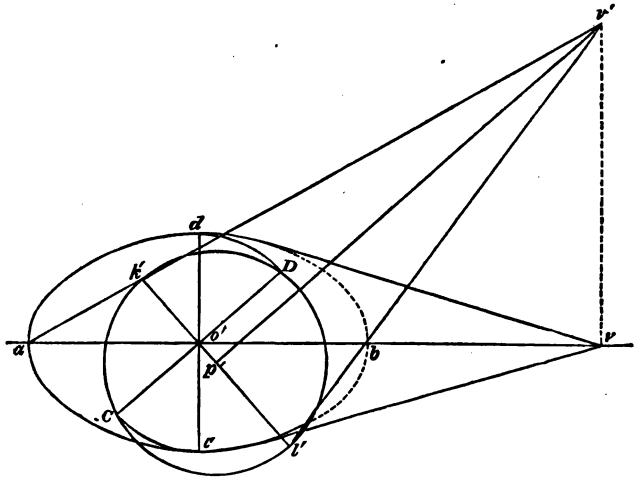
The curves in which the cones intersect the planes of projection are called its traces.

A Right Conic surface is one which can be represented by a circle as its plane directrix, and a point for its vertex in the perpendicular to the plane of the circle at its centre.

A Cone is given when its *vertex* and a *trace* are given.

1. *Given* ab, *the major axis of the ellipse which forms the oblique section of a right cone by the horizontal plane, and* vv', *the projections of the vertex, to draw the plan of the frustum.* Fig. 28.

Fig. 28.



Since the plan *v* in *ab* produced is given, it will be obvious that when the ellipse is obtained, the required plan will be completed by drawing tangents to the ellipse through *v*. Hence the problem consists chiefly in finding *cd*, the

minor axis of the curve. This line is the horizontal trace of the plane of a circular section of the cone made through point *o'*, the bisection of *ab*, and forms a chord of that circle.

This circle being perpendicular to the axis, *VP*, of the cone, is inclined to the horizontal plane at the complement of the angle of inclination of that axis. The circle must therefore be rabatted and its chord shown.

Take a vertical plane through *V* and *ab*; and on it draw *v'a*, *v'b*, the traces of the conic surface. Through *o'* draw *k'l'* making equal angles with these lines; this is the vertical trace, and elevation of the circular base.

Rabat this circle into the vertical plane as shown in Fig. 28, about *k'l'*, its vertical trace and diameter. Draw through *o'* the chord *CD* at right angles to *k'l'*. *CD* gives the length *cd* of the required minor axis.

2. *To determine the projections*, pp', *of a point*, P, *on the surface of a cone.*

Assume *p*, the plan of the point, and through *p* draw a plan of the generatrix of the cone and show its elevation. The point *p'*, in which the projecting line from the plan *p* cuts the elevation of the generatrix, will be the elevation of *P*.

3. *A cone with vertex* 5 *in. high, axis inclined* 50°, *and in plan making an angle of* 30° *with the ground line, envelops a sphere of* 1 *inch radius and centre* 2 *in. high. Determine horizontal trace of cone and circle of contact.*

As in Problem 4 on "The Cylinder," use the plan of the axis as the ground line for an auxiliary elevation. In this auxiliary plane set up the elevation of the axis by drawing a line inclined 50° with *xy* through the point which is the elevation of the centre of the sphere, and determine *vv'*, the

vertex of the cone, 5 in. high. Tangents from v' to the circle representing the elevation of the sphere will mark off on xy the line which is the elevation of the horizontal trace of the cone. This same line is also the length of the major axis of the ellipse which is the horizontal trace of the cone.

To find the minor axis of the same ellipse, the point must be determined, on the circle of contact, through which a line from the extremity of the minor axis to the vertex would pass. In the auxiliary elevation this line will be shown by joining the vertex and the centre of the major axis. The intersection of this line with the straight line representing the elevation of the circle of contact gives the elevation of the point required, through the *plan* of which point a line drawn from the plan of the vertex marks off the extremity of the minor axis sought.

See also "Note" 1, Prob. 4, on "The Cylinder."

NOTE. The straight line which, in the auxiliary elevation, will be the elevation of the circle of contact, will not pass, as in the case of the enveloping cylinder, through the elevation of the centre of the sphere. It is the line joining the tangent points before determined.

The plan and new elevation of the circle of contact can now be readily determined. The student should show the *traces* of the plane of the circle of contact.

The horizontal trace of the cone would represent the shadow of the sphere, cast upon the H.P. by rays of light proceeding from the fixed point V; and the circles of contact would mark the boundary of the light and shade.

4. *The centre,* O, *of a sphere of* 1 *in. radius, is* 1·5 *in. above the horizontal plane, and* 2·5 *in. in front of the vertical plane. A straight line in the horizontal plane,* 2·25 *in. from*

the plan, o, *of the centre of the sphere, and making an angle of* 30° *with* xy, *is the plan of the axis, inclined at* 50°, *of a right cone whose convex surface touches the sphere. The vertex of the cone is* 5 *in. high, and its circular base touches the horizontal plane. Determine its plan and elevation and the point of contact of the surfaces.*

(1) Find the projections vv' of the vertex V.

(2) Determine a plane containing the centre O of the sphere and the axis, as in Prob. 6 on "The Cylinder."

(3) Rabat this plane with its contained great circle and axis. Let V' be the rabatted vertex. From V' draw a line touching the rabatted great circle in P'. The angle between the rabatted axis and this tangent line is the semi-vertical angle of the cone. When this angle is found, the projections of the cone are easily determined by the aid of an auxiliary vertical plane of elevation containing the axis.

(4) The projections pp' of the point of contact P are at once determinable from the rabatted point P'.

The "Note" to Prob. 6 on "The Cylinder" applies, *mutatis mutandis*, to this.

5. *Refer to Prob.* 4 *on "The Sphere." A point* 1·5 *in. distant in plan from the centre of the sphere and* 4·5 *in. high, is the vertex,* V, *of a conic surface of which the traced curve is the directrix; find its horizontal trace.*

It is only necessary to draw a number of generatrices from V through points in the curve, the line joining the horizontal traces of these generatrices will be the required trace of the conic surface.

6. *A right cone revolves uniformly about its axis fixed in position, and a curve is traced upon its surface by a point*

moving with uniform velocity in a straight line from the vertex to the base of the cone in a fixed plane containing the axis. Determine the projection of the curve when the rotation of the cone and the motion of the point are so timed, that the point moves from the vertex to the base while the cone is making one revolution. Radius of base 1·5 *in.; axis* 3·5.

The student after working Prob. 4 on "The Sphere," and Prob. 7 on "The Cylinder," will find this problem and its development an easy exercise. It may be interesting to note that this curve is Hogarth's famous "line of beauty."

The gradient of this spiral depends entirely upon the relative velocity of the tracing point and the cone.

7. *A semicircle is described upon a generatrix of the cone in the last problem as a diameter, and the plane containing it is wrapped round the conic surface. Draw the projections of the resulting curve.*

Develop the cone and describe the semicircle on a generatrix in the development. If the segment of the circle which is the development of the base of the cone be divided into any number of equal parts, and the circular base of the solid be divided into the same number of equal parts, a set of generatrices in the development may be drawn of which the corresponding projections are easily obtainable. The points in which certain of the generatrices in the development cut the semicircle may now have their corresponding projections found, in the projections of these generatrices, by proportionate division, and the curve be drawn through them in plan and elevation. Similarly any plane curve may be wrapped round any developable surface whatever, or the shortest line between any two given points be traced on the surface. For an example of the latter problem, see No. 8 on "The Cylinder."

TANGENT-PLANES AND NORMALS TO CURVED SURFACES.

General Remarks.

DEF. The tangent line to a curve at any point P is the limiting position a straight line passing through P, and cutting the curve in another point Q, assumes when the said straight line is turned about P in such a manner that Q continually approaches P and ultimately coincides with it.

The tangent line to a curve at any point is also a tangent, at the same point, to any surface on which the curve is traced.

DEF. The tangent-plane to a curved surface at any point P is the plane containing two tangent lines, at the same point P, to any two curves traced upon the surface through that point.

THEOREM. The tangent-plane to a curved surface at any point P contains all the tangent lines drawn at the same point to all the lines that can be traced upon the surface at that point.

COR. If one, or more, straight lines can be traced on a curved surface at a point P, the tangent-plane to the surface, at that point, meets the surface in the said line or lines.

Thus, the tangent-plane to a Ruled Surface at any point meets the surface along a generatrix drawn through that point. When the surface is Developable, *e.g.* the cone and cylinder—the tangent-plane *at* a point is also a tangent-plane *along a generatrix.*

DEF. The normal to a curved surface at any point P is the perpendicular drawn through P to the tangent-plane to the surface at that point.

Only one normal can be drawn to a surface at the same point.

All the normals to a *cone* or a *cylinder* (or other Developable surface) along a generatrix lie in one plane perpendicular to the tangent-plane which touches the surface along the same generatrix. This is not true of Twisted or Skew surfaces.

Two curved surfaces touch at a point when the tangent-plane to one of the surfaces at that point is also a tangent-plane to the other.

Two lines that meet completely determine the plane containing them. From this and the definition of a tangent-plane given above, it follows that, in general, the problem of constructing a tangent-plane to a surface is reducible to that of drawing two tangent lines to any two lines that can be conveniently traced on the surface at the point of contact of the tangent-plane. When the surface is a *Ruled* one the problem is further simplified by one at least of the lines traceable on the surface through the point of contact being straight. Sometimes, as for example in the hyperboloid of one sheet and the hyperbolic paraboloid, two such straight lines can be drawn.

When the normal to the required tangent-plane is easily determinable, it may be first drawn and the tangent-plane subsequently taken at right angles to it. Occasionally, auxiliary surfaces, having the same tangent-plane as the given surface, may be advantageously assumed. Thus the following theorem is frequently required:—

THEOREM. The tangent-plane to a cone or cylindrical surface is a tangent-plane to the spheres or surfaces of revolution enveloped by it.

It was pointed out when treating of the cylinder, that any curved line in space might be regarded as the intersection of two surfaces, consequently the line of intersection of two tangent-planes, one to each surface, at the common point, *P*, would be a tangent line to the curve at the same point. This tangent line is obviously, therefore, perpendicular to the plane containing the two normals to the surfaces. Thus, if *P* were a point in the intersection of two spheres of centres *X* and *Y*, a line through *P* perpendicular to the plane of *XYP* would be a tangent line to the curve of intersection at the point *P*.

PROBLEMS ON TANGENT-PLANES TO CURVED SURFACES.

In the foregoing problems on curved surfaces it was found convenient to treat the sphere first, and to follow with the cylinder and cone. In the treatment of Tangent-Planes to these surfaces this order can be inverted with considerable advantage.

TANGENT-PLANES TO CONES.

1. *A right cone, base 2 in. diameter, and axis 4 in. long, is so placed that the circular base touches the horizontal plane; the axis is inclined at* 50°, *and its plan makes an angle of* 30° *with* xy. *Determine a point* P *on the conic surface and draw a tangent-plane to the cone at that point.*

Find the horizontal trace of the cone. Draw the generatrix *VP*, and find its horizontal trace. This will be a point in the ellipse which is the horizontal trace of the surface. Through this point draw a tangent line to the ellipse for the H.T. of the tangent-plane.

The point in which the H.T. meets *xy* will be one point in the V.T.; another will be the vertical trace of a

line drawn from *V* parallel to H.T. Through these two points the vertical trace (V.T.) of the tangent-plane can be drawn.

NOTE. The plane containing the generatrix *VP* and the axis of the cone, contains all the normals to the tangent-plane, and is at right angles to it. Therefore, if a line be drawn from any point in the generatrix (as *V*) at right angles to the plane of the normals its horizontal trace will be a point in the H.T. of the tangent-plane.

This enables us to find the H.T. of the tangent-plane without depending on the plane geometrical problem for drawing a tangent line to a conic section.

2. *Take the same cone as in the last problem, and assume a point* P *outside the conic surface. Draw tangent-plane as before, and the line of contact.*

α. Find the horizontal trace of the conic surface, and the horizontal trace of the line *VP*. The H.T. of the tangent-plane will be the line drawn from the horizontal trace of the line *VP* to touch the horizontal trace of the conic surface. Two such lines may be drawn, *i.e.* two tangent-planes, each satisfying the conditions, are possible. The V.T.s may be found as in the last problem.

β. When the horizontal trace of *VP* is inaccessible, the point *P*, or any convenient point in the line *VP*, may be made the vertex of a similar cone, and the two H.T.s drawn touching the two horizontal traces of the conic surfaces. In the particular case in which the line *VP* is parallel to the horizontal plane, the H.T.s will be tangents drawn to the horizontal trace of the conic surface parallel to the plan of the line *VP*.

The *lines of contact* will be the generatrices of the cone drawn from the points in which the H.T.s of the tangent-planes touch the horizontal trace of the surface.

3. *Take the same cone as before and any line inclined at θ° and ϕ° to the horizontal and vertical planes respectively. Draw a tangent-plane to the cone parallel to this line.*

Draw a line from V parallel to the given line and find its horizontal trace. The H.T.s of the tangent-planes which satisfy the conditions can then be drawn from this point tangents to the horizontal trace of the surface.

Note the limits. When the line from the vertex parallel to the given line falls inside the conic surface the problem is impossible.

The above solution enables us to draw a tangent-plane to a cone perpendicular to a given plane: such a tangent-plane being of course parallel to any line taken at right angles to the given plane.

4. *To the same cone as before draw a tangent-plane which shall be inclined at 70° to the horizontal plane.*

THEOREM. A common tangent-plane can always be drawn to two cones having the same vertex, unless one of the sheets of one of the conic surfaces lies wholly within the surface of the other without touching it.

The number of tangent-planes possible depends upon the relation between the two cones, and may be one, two, three, or four.

To work the problem, make V, the vertex of the given cone, the vertex also of a right cone standing on the horizontal plane, with generatrices inclined at 70°.

Any line that can be drawn a tangent to both the horizontal traces of the cones will be a H.T. of a tangent-plane satisfying the requirements of the question; its V.T. can be found as before.

Limits. The given angle of inclination of the tangent-plane must lie between 90° and $\alpha°$, where $\alpha°$ is the angle which the generatrix of minimum inclination makes with the horizontal plane.

5. *Given an oblique plane and a straight line* VO *meeting it in the point* V, *determine a right cone that shall touch the given plane and have the straight line* VO *for its axis.*

(1) Construct the plane of the base at right angles to the line *VO* through the point *O*, and find the intersection of this plane with the given plane.

(2) Rabat the plane of the base, with the point *O* and the line of intersection, into one of the planes of projection, and determine the radius of the circle having *O* for its centre and the intersection line for its tangent. This will be the radius of the base of the cone.

(3) Take an auxiliary vertical plane containing *VO*, and thence complete the projections of the cone.

6. *Given a right cone, axis oblique to both planes of projection, and a point* P *in space. Determine a second cone having the point* P *for its vertex and a given straight line passing through* P *for its axis that shall touch the given cone.*

The two cones will have for their common tangent-plane, a plane passing through their vertices.

Draw, therefore, a tangent-plane to the given cone through the given point *P*, and determine by the last

problem a right cone touching this plane and having the given straight line for its axis.

NOTE. Two tangent-planes can be drawn to the given cone through the given point, hence two cones can be found satisfying the conditions of the problem.

7. GENERAL CASE. *To draw a normal to a conic surface through a given point not in the surface.*

Find the curve which is the locus of the feet of all the perpendiculars let fall from the given point upon all the tangent-planes to the cone. The point, or points, in which this curve meets the surface will determine the normal or normals.

To find the intersection of this curve with the given surface a second cone may be taken, having the same vertex as the first and the curve for its directrix. Lines from the point, or points, in which the horizontal traces of these cones meet, to the common vertex, will determine the intersection of the curve and the given surface.

8. GENERAL CASE. *To draw a normal to a given cone parallel to a given plane.*

Draw a tangent-plane to the given cone perpendicular to the given plane (see Prob. 3). Any perpendicular to this tangent-plane from a point in its line of contact with the surface will be *normal* to the given cone, and parallel to the given plane.

TANGENT-PLANES TO CYLINDERS.

1. *A right cylinder, base two inches diameter and axis four inches long, is so placed that the circular base touches the horizontal plane; the axis is inclined at* 50° *and its plan makes an*

angle of 30° *with* xy. *Determine a point*, P, *on its surface, and draw a tangent-plane to the cylinder at that point.*

Draw a generatrix of the cylinder through *P*, and find its horizontal trace in that of the cylinder. The tangent to the horizontal trace of the cylinder at the point which is the horizontal trace of the generatrix will be the H.T. of the required tangent-plane, the V.T. of which can then be found.

The generatrix through *P* is the line of contact of the tangent-plane with the surface. The plane through this and the axis contains all the normals to the tangent-plane. Hence the latter might be determined at right angles to the plane of the normals as in the parallel case of the right cone.

2. *Same cylinder as in the last problem, and a point*, P, *without the surface. Draw tangent-plane and show line of contact.*

Through *P* draw a line parallel to the generatrices of the cylinder. The H.T. of the tangent-plane can be drawn through the horizontal trace of this line to touch the horizontal trace of the surface. Find V.T. and the line of contact as in other problems and compare with the parallel problem on the cone.

3. *Same cylinder as before, and a line inclined* θ° *and* ϕ° *to the horizontal and vertical planes respectively. Draw tangent-plane parallel to the given line.*

Determine a plane containing the given line and parallel to a generatrix of the cylinder. A straight line, touching the horizontal trace of the cylinder, and parallel to the horizontal trace of the plane determined above, will be the H.T. of the required tangent-plane, whence the line of contact and the V.T. may be found.

Note the limits. The angle θ^0 must lie between 90° and the angle of inclination of the generatrices of the cylinder.

This problem enables us to determine a tangent-plane to a cylinder perpendicular to a given plane.

4. *Draw a plane inclined* 70° *to the horizontal plane and touching the cylinder used in the foregoing problems.*

Determine any right cone standing on its circular base having its generatrices inclined at 70°.

Find the horizontal trace of a tangent-plane to this cone parallel to a generatrix of the given cylinder. A line parallel to this horizontal trace touching the horizontal trace of the cylinder will be the H.T. of the required tangent-plane. Complete by drawing line of contact and V.T.

Limits. The given angle must lie between 90° and the angle of inclination of the generatrices of the given cylinder.

5. *Given two points,* B *and* C, *at equal distances from a given plane, as the foci of an elliptic section of a right cylinder which touches the plane: determine the cylinder and its line of contact with the plane.*

The plane of the ellipse is perpendicular to the given plane, and the curve touches the given plane at the extremity, Q, of its minor axis. This point may be determined by a perpendicular to the plane through O, the point of bisection of BC. The line BC being parallel to the given plane, the foot of this perpendicular from O to the plane will be Q.

The major axis will be equal to $BQ + CQ$, whence the ellipse is completely determined.

The position of a circular section of the cylinder perpendicular to the given plane, with radius equal to OQ, can be determined, with one extremity of a diameter coinciding with an extremity of the major axis of the ellipse. See the construction given in Prob. 1 on "The Cylinder."

The straight line through Q and the point in which the circle touches the plane, is the generatrix in which the cylinder touches the plane.

Work the problem:

(1) When the given plane is horizontal.

(2) " " " oblique.

6. *To determine a right cylinder which shall have a given straight line for its axis and shall touch a given right cylinder.*

It is obvious that the two cylindrical surfaces will have a common tangent-plane at their point of contact. This point, which lies in a straight line at right angles to both axes, must therefore be determined.

Let AB be the axis of the given cylinder, and CD that of the required one: AB not lying in the same plane with CD. Determine PQ the common perpendicular to these axes by Problem 23, Chap. III., meeting AB in point P, and CD in Q.

Find T the point in which PQ pierces the surface of the given cylinder. T is the point of contact, and QT the radius of the circular section of the required cylinder, which may hence be constructed. See Problem 5 on "The Cylinder" for special exercise.

Another Method.

Assume any point, P, and draw through it parallels to the two given axes. Determine the horizontal traces of

these lines and join them for the horizontal trace of the plane containing them. This plane will be parallel to the generatrices of both cylinders.

Another plane parallel to this, and touching the given cylinder, will be the common tangent-plane of the two cylinders, and its horizontal trace will, therefore, be a tangent line to the horizontal traces of both surfaces.

The horizontal trace of the required cylinder may be drawn in either of the following ways:—

(1) Find the horizontal trace of its given axis and the real distance from any point *M* in this axis to the tangent plane. This will be the radius of the contained sphere. The cylinder enveloping this sphere and having the given axis can be drawn by Problem 4 on "The Cylinder" and its horizontal trace determined. *Or*,

(2) Draw from any point in the given axis a perpendicular to the tangent-plane, and find the point in which the perpendicular meets the plane. A line through this point parallel to the given axis will be the line of contact of the tangent-plane with the required cylinder, and the H.T. of this line will be a point in the H.T. of the cylinder. The true length of the perpendicular to the plane will be the semi-minor axis of the elliptical trace. On drawing the minor axis of the ellipse through the horizontal trace of the given axis of the cylinder at right angles to the plan thereof, we have one axis and a point in the curve, from which data the ellipse can be drawn, by means of a strip of paper, on the well-known principle of the trammel.

The generatrices drawn from the two points in which the ellipses are touched by the horizontal trace of the

tangent plane will meet in T, the point of contact of the cylinders.

7. *Given a right cylinder, axis oblique, a point in space and a straight line passing through it; determine a right cone that shall have the given line for its axis, the point for its vertex and touch the given cylinder.*

Refer to the parallel problem on "Tangent-Planes to Cones," Problem 6, for method.

8. GENERAL CASE. *To draw a normal to a cylinder through a point not in the surface.*

Construct the curve which is the locus of the feet of all the perpendiculars let fall from the given point upon all the tangent-planes to the cylinder, as for the parallel case on the cone. The intersection of this curve with the cylinder will determine the normal or normals.

This intersection may be found by making the curve the directrix of a new cylinder whose generatrices are parallel to those of the given cylinder. The common generatrices of these surfaces, drawn from the points in which their horizontal traces meet, will determine the intersections of the curve with the given surface, and, therefore, the normals, if there are more than one.

9. GENERAL CASE. *To draw a normal to a given cylinder, parallel to a given plane.*

Draw a tangent-plane to the given cylinder perpendicular to the given plane (Prob. 3).

Any perpendicular to this tangent-plane from a point in its line of contact with the cylinder will be *normal* to the given surface, and parallel to the given plane.

10. *Determine a helix of* 1·5 *in. pitch on the surface of a right cylinder, axis vertical, diameter of base* 2 *in., as in*

Problem 7 on "The Cylinder," and draw the curve in which the surface generated by the motion of the tangent line to the helix intersects the horizontal plane.

Let P be any point on the helix. Its elevation p' will be a point in the elevation of that curve, and its plan p a point in the circumference of the circle which is the plan of the cylinder.

A tangent line to this circle at the point p will be the plan of the tangent line to the helix at the point P in the curve, and as this line is inclined at $\alpha°$ (the *angle of the screw*) to the horizontal plane, its horizontal trace may be found for one point in the required curve.

Similarly other points may be found, and the curve, which is the involute of the circle forming the base of the cylinder, drawn through them.

NOTE. This surface is a "Developable Helixoid" of which the helix is the *Edge of Regression.*

TANGENT-PLANES TO SPHERES.

1. *A point,* O, 2 *in., and* 1·5 *in., from the horizontal and vertical planes respectively, is the centre of a sphere of* 1 *in. radius. Draw the sphere, and a tangent-plane to it at a point* P *on its surface,* 2·3 *in. from H.P., and* 1·8 *in. from V.P.*

To find P, take an auxiliary vertical plane parallel to the vertical plane of projection, 1·8 in. from it, cutting the sphere. The elevation of the section of the sphere by this plane will be a circle shown in true shape. A point in the elevation of the circle 2·3 in. above xy, will be p' the elevation of the point P, whence the plan p can be readily found.

If any two planes be assumed passing through P and the centre O of the sphere, they will cut the spherical

surface in two great circles having a common radius *OP*, and since the tangent lines at *P* to both of these circles are at right angles to this radius, the tangent-plane to the sphere at the same point is also perpendicular to *OP*; *i. e.* *OP* is the *normal* to the tangent-plane. Hence,

To draw a tangent-plane to the sphere at the point *P*, it is only necessary to draw a plane at right angles to *OP* to pass through the point *P*, by Prob. 13, Chap. III. *Converse.*

NOTE. The *normal* to a sphere from any point whatever is obviously the straight line drawn from the point to the centre of the sphere.

2. *Determine a tangent-plane to the sphere in the last problem containing a given point*, P, *without the spherical surface. Let* P *be in the vertical plane*, 5 *in. high and* 3·75 *in. from the elevation* o′ *of the centre of the sphere.*

Make *P* the vertex of a cone enveloping the sphere (Prob. 3 on "The Cone"). Any tangent line to the horizontal trace of the cone will be the horizontal trace of a plane fulfilling the required conditions.

In cases in which the horizontal trace of the enveloping cone is inaccessible the *circle of contact* may be determined, and the tangent-plane to the sphere drawn at any assumed point in this circle, by the last problem.

NOTE. There is no limit to the number of solutions possible.

3. *Take the sphere and point* P *in Prob.* 2, *and assume another point*, Q, *also without the spherical surface. Determine a tangent-plane to the sphere that shall pass through* P *and* Q.

NOTE. Since P and Q lie in the tangent-plane, the straight line PQ indefinitely produced will also lie in the tangent-plane. The tangent-plane to the sphere through P and Q will therefore be a tangent-plane to the sphere through every point in the straight line PQ produced; and if this line do not meet the spherical surface two planes touching the sphere can be drawn through it; if the line touch the sphere one only can be drawn, and if it pass through the spherical surface the problem is impossible.

First Method. Make P and Q, or any two other convenient points in the line PQ produced, the vertices of two cones enveloping the sphere. If the horizontal traces of these cones are determinable, a line touching them may be taken for the horizontal trace of the tangent-plane required; but if not, determine the circles of contact of the two cones and the points R and S in which they intersect, and draw the traces of the planes passing through the points PQR and PQS, for those of the tangent-planes satisfying the conditions. *Or*, tangent-planes to the sphere at the points R and S may be drawn as in Prob. 1.

To shorten the work, one cone may be assumed with its vertex in PQ at the same level as the centre, O, of the sphere, the plan of the circle of contact being in this case a straight line.

Second Method. Take a plane through the centre, O, of the sphere at right angles to the straight line joining the points P and Q. Let I be the intersection of this line and plane. Straight lines, through the point I, touching the great circle in which this plane cuts the sphere, give the points R and S, whence the plane may be determined as before.

Third Method. Find the horizontal trace of the line PQ, produced if necessary, and the horizontal trace of a cone having its vertex in this line. The two tangent lines to the conic trace, through the H.T. of the line PQ, will be the horizontal traces of the two tangent-planes. The vertical trace of the line will also be a point in the vertical traces of the planes, which may be thus determined.

Fourth Method. Draw a cylinder enveloping the sphere with its axis parallel to PQ. The two tangent-lines to the horizontal trace of the enveloping cylinder, through the H.T. of the line PQ, will be the horizontal traces of the tangent planes.

4. *Take the sphere, and point* P, *as in Prob.* 2, *and determine a tangent-plane to the sphere that shall pass through* P *and be inclined to the horizontal plane at an angle of* 75°.

First Method. Make the given point P the vertex of a cone enveloping the sphere, and draw a tangent-plane to this cone at 75° to the horizontal plane, by Prob. 4 on "Tangent-Planes to Cones."

Second Method. Envelop the sphere by a cone with its axis vertical and generatrix inclined at 75°.

A plane containing P and touching the cone will satisfy the conditions. "Tangent-Planes to Cones," Prob. 2.

5. *The centre,* O, *of a sphere of* 1 *in. radius is* 1·5 *in. from each plane of projection. Draw the traces of a plane inclined at* 75° *and* 60° *to the horizontal and vertical planes respectively, and touching the sphere.*

Determine a right cone, generatrix inclined 75° to the horizontal plane, enveloping the sphere.

Similarly, find the vertex, V, of another right cone, generatrix inclined 60° to the vertical plane, enveloping the sphere.

The tangent-plane to the first cone through the point *V*, the vertex of the latter, satisfies the given conditions.

NOTE. Four planes—two parallel pairs—are possible. Compare this problem with Prob. 7, Chap. III.

6. *Determine the sphere, and the point* P, *as in Prob. 2. Draw the traces of a tangent-plane to the sphere that shall pass through* P *and be parallel to a line inclined at* 60° *to the horizontal plane and* 30° *to the vertical.*

Envelop the sphere by a cone having *P* for its vertex, and determine a tangent-plane to this cone parallel to the given line. "Tangent-Planes to Cone," Prob. 3.

NOTE. Refer to Problem 3, "Tangent-Planes to Sphere," and compare.

7. *Determine a tangent-plane to two spheres of unequal sizes, unequal heights, and free from one another, which shall*

(α) *contain a given point* P,

(β) *be parallel to a given straight line.*

Envelop the two spheres by a cone.

NOTE. Two enveloping cones are possible. In one case both spheres will be enveloped by the same sheet of the conic surface, and in the other case, one sphere will be enveloped by one sheet of the cone and the other sphere by the other sheet.

For (α). Determine a tangent-plane to the enveloping cone that shall contain *P*. *Or*, Make *P* the common vertex of two cones—one enveloping one sphere and the other the other sphere—and determine the tangent-plane to them.

For (β). Determine a tangent-plane to the enveloping cone parallel to the given line.

8. *Three equal spheres, diameters 2 in. each, have their centres at the angles of a vertical equilateral triangle of 4 in. side, with one side horizontal. Draw a tangent-plane to the three spheres:—*

(α) *To pass above the two lower and under the upper sphere.*

(β) *To pass above one of the lower and under the other two spheres.*

For (α). Draw an auxiliary elevation on a vertical plane at right angles to the plane of the triangle. A tangent to the two circles which are the elevations of the three spheres can be drawn for the auxiliary vertical trace of the required plane. The H.T. will of course be perpendicular to the auxiliary *xy*.

For (β). Make an elevation on a vertical plane parallel to the plane of the triangle.

Determine a cylinder containing the upper and one of the lower spheres, and a second cylinder parallel to the first cylinder and containing the third sphere.

The horizontal trace of the required plane can now be drawn as a tangent to the two ellipses which are the horizontal traces of the two cylinders.

The tangent lines on the cylinders can be shown by drawing generatrices from the tangent points of the ellipses.

Find the points of contact with the spheres by drawing perpendiculars to the tangent-plane from their centres. These points are fixed where the projections of the perpendiculars cut the generatrices which are the lines of contact with the cylinders.

9. *Draw the three spheres touching in Problem 2 on "The Sphere" and determine a tangent-plane to the three.*

NOTE. The horizontal plane of projection is one tangent-plane.

To find the other. Determine plan and elevation of the axes of two cones, one enveloping *A* and *B*, and the other enveloping *A* and *C*. Lines through the centres of the spheres give these axes. The horizontal traces of these axes will be the vertices of the enveloping cones, and will also be points in the H.T. of the tangent-plane required. The vertical trace will be most readily found by the aid of an auxiliary vertical plane taken at right angles to the H.T. of the tangent-plane.

10. *Determine the tangent-planes to three spheres* A, B, C, *of unequal sizes, unequal heights, and free from one another.*

Determine two cones enveloping *A* and *B*, and *A* and *C*, respectively. A tangent-plane to one of these cones containing the vertex of the other will satisfy the conditions. *Or*, the vertex of one of the cones enveloping two of the spheres may be made the vertex of a second cone enveloping the third sphere, and the tangent plane to these cones determined for the tangent-plane required. The student will here see that this affords a method of determining a tangent-plane to a given cone and a given sphere.

Refer to "Note," Problem 7 above, and observe that eight tangent-planes to the three spheres can be found.

The student can exercise himself by showing that the vertices of the six possible enveloping cones are in four straight lines—three vertices in each line.

11. *Determine the cone, Prob.* 1. "*Tangent-Planes to Cones*," *and a sphere* ·75 *in. radius touching the horizontal plane in a point*, P, *assumed at pleasure. Show the traces of a tangent-plane to the sphere and the cone.*

One solution of this problem was referred to in No. 10 above.

Another Method. Envelop the sphere by a cone similar to the given one, and determine a tangent-plane to one of these cones and containing the vertex of the other. This will be the plane required.

12. *To draw a normal to a sphere parallel to a given straight line.*

A line through the centre of the sphere parallel to the given line will be the normal required.

13. *To draw a tangent-plane to a sphere parallel to a given plane.*

Draw a perpendicular from the centre of the sphere to the given plane. This will be the normal to the required tangent-plane, which may at once be drawn at right angles to the normal through the point in which the latter meets the surface.

Problems on Surfaces of Revolution.

Having already defined a surface of revolution in the general remarks at the head of this chapter, it will be only needful here to summarise the chief properties in the following theorems:—

Theorem I. All sections of a surface of revolution by planes at right angles to the axis, are circles having for their centres the points in which the axis meets the section plane.

Theorem II. All meridian sections of a surface of revolution are similar and equal curves.

Theorem III. All the tangent lines drawn from a fixed point in the produced axis of a surface of revolution

to meridian sections, touch the surface in points which lie on the circumference of a circle whose plane is at right angles to the axis. That is, the *locus* of all tangent lines to a surface of revolution drawn from a fixed point in the produced axis, is a right cone.

THEOREM IV. The tangent line to any circular section of a surface of revolution at a point in it, is perpendicular to the meridian section of the surface through that point.

COR. 1. The tangent-plane to the surface at any point is perpendicular to the meridian plane through the same point.

COR. 2. The normal to a surface of revolution *at* any point, lies wholly in the meridian plane passing through that point, and is normal to the meridian section at the same point.

COR. 3. The normal *from* a given point to a surface of revolution is the normal to the section of the surface made by a meridian plane containing the given point.

COR. 4. The tangent-planes to a surface of revolution at different points in the same circular section pass through the same point in the axis.

COR. 5. The normals to a surface of revolution at different points in the same circular section meet the axis at the same point.

A surface of revolution is given when its axis and its plane generatrix (meridian section) are given. But, whatever line be given as the generatrix, it can always be reduced to an equivalent plane generator.

The *right cone*, and the *right cylinder*, may be regarded as surfaces of revolution, but in the general treatment of conic

and cylindrical surfaces it is necessary to consider them in another point of view. This has been already done as far as the limits of this book allow. The sphere, which is essentially a Surface of Revolution, has also for convenience been already separately considered.

The surface generated by an ellipse revolving about either of its axes is called a *Spheroid.* It is also a particular case of an ellipsoid, namely an *Ellipsoid of Revolution.* It is *prolate* or *oblate* according as it revolves about the major or minor axis.

A few problems of a general character here follow:—

1. *An ellipse, major axis* 2·75 *in., minor axis* 2 *in., generates a surface by revolving about its major axis. Show plan and elevation of the surface when the axis of revolution is vertical, and find the projections of a point,* P, *on it.*

The plan of the surface will be simply a circle of 2 in. diameter, and its elevation, an ellipse of the dimensions given in the question.

To find pp', the projections of the point P. If the plan p be given, assume a meridian section to be made through P and to be turned round, with the point P on it, till parallel to the vertical plane. The height of P may then be found by drawing a projecting line from the new position of p in plan to cut the ellipse which is the elevation of the surface. Either of the two points in which, in general, the projecting line meets the ellipse, may be the height of P, according as we assume the point to be on the upper or lower part of the surface.

If p' be given, p may be found by taking a horizontal section through P, the plan of which section will be a circle

passing through the plan p. This point may then be readily determined by drawing a projecting line from p' to meet the circle.

2. *To determine a cone enveloping the spheroid in the last problem, when the vertex is a point in the produced axis of the surface.*

Let v' be the elevation of the vertex. Draw from v' a tangent line to the ellipse which is the elevation of the spheroid and call the point of contact t'. The line $v't'$ is the elevation of a generatrix of the enveloping cone, and its plan will be a line drawn from v parallel to xy. The circle of contact (the directrix) will be the circle in which a horizontal section plane through T cuts the surface, and the horizontal trace, or base of the cone, will be a circle passing through the H.T. of the generatrix VT, and concentric with the circle which is the plan of the spheroid.

In a similar manner, any number of enveloping cones can be drawn by taking other points in the axis for vertices.

3. *Determine the spheroid as before, and assume a point,* P, *in space, without the surface, and not in the axis. Find the cone which has* P *for its vertex and envelops the spheroid.*

(1) Determine any cone, as in the last problem, enveloping the spheroid, and having its vertex in the axis of revolution produced.

(2) Draw a tangent-plane to this cone passing through P. This will also be a tangent-plane to the spheroid, and the point in which the circle of contact of the cone with the spheroid meets the line of contact of the tangent-plane with the same cone, will be the point of contact of the tangent-plane with the spheroid. Two tangent-planes being

possible to the enveloping cone, there will, of course, be two such points.

(3) Through these points of contact draw two lines to the point P. Since these lines lie in tangent-planes to the spheroid, and contain the points of contact of the tangent-planes, they are tangents to its surface, and, inasmuch as they pass through P, they are generatrices of the required enveloping cone.

Similarly, by taking a series of other points in the produced axis for the vertices of new cones, any required number of pairs of generatrices may be found, and their horizontal traces joined by a curve for the H.T. of the enveloping cone. The curve joining the points of contact of the tangent-planes will give the curve of contact of the enveloping cone with the spheroid.

4. *Determine a cylinder enveloping the spheroid in Problem* 1, *when the generatrices are parallel to a given line inclined at* 60° *to the horizontal plane, and in plan making an angle of* 30° *with* xy.

(1) Envelop the spheroid by any cone, as in Problem 2, having its vertex in the produced axis.

(2) Draw a tangent-plane to this cone parallel to the given line. This will also be a tangent-plane to the spheroid, and its point of contact may be found as in the last problem. Here too, since there are two tangent-planes, there will be a pair of points of contact.

(3) Through this pair of points, two lines, parallel to the given line, may be drawn for a pair of generatrices of the enveloping cylinder.

Similarly, any number of pairs of generatrices can be found, and the enveloping cylinder, with its curve of contact, completely determined.

NOTE. The last two problems might be worked by the aid of auxiliary sections of the surface by planes which pass, in Prob. 3, through the given point P, and, in Prob. 4, by planes taken parallel to the given line. Tangent lines to the sections give pairs of generatrices of the enveloping surfaces.

5. *A surface is generated by the revolution of a circle of* 1 *in. diameter about a fixed straight line in the plane of the circle and* 1·5 *in. from its centre. Determine a point,* P, *on its surface, and the enveloping cones and cylinder as in Probs.* 1, 2, 3, *and* 4 *above.*

This surface, which is an *anchor ring*, or *annulus* of a circular section, will afford the student an additional exercise in the application of the methods used in the solution of the problems mentioned, without presenting any special features of difficulty.

TANGENT-PLANES AND NORMALS TO SURFACES OF REVOLUTION.

After what has been already given, it will be only necessary to speak of the following problems in the most general terms.

1. *To determine a tangent-plane to a surface of revolution at a given point on it.*

Draw the normal (Theorem IV. Cor. 2, above) and the tangent-plane at right angles to it, through the given point.

2. *To determine a tangent-plane to a surface of revolution that shall contain a given straight line or its equivalent, two points in space.*

The student on referring to the parallel problem, "Tangent-Planes to Spheres," will not be at a loss for several possible solutions, the easiest of which is to envelop the surface by a cylinder with generatrices parallel to the given line. A tangent-plane to this cylinder through any point in the line will satisfy the conditions.

3. *Determine a tangent-plane to a surface of revolution that shall make an angle θ^0 with H.P. and contain a given point,* P, *in space.*

Make *P* the vertex of a cone enveloping the surface of revolution, and determine a tangent-plane to this cone that shall make an angle of θ^0 with H.P. This will be the plane required.

4. *To draw a normal to a surface of revolution parallel to a given straight line.*

Determine a meridian section of the surface by a plane parallel to the given straight line, and a tangent line to the curve at right angles to the projection of this line upon the plane. A perpendicular to the tangent line, through the point of contact, will be the required normal.

5. *To draw a tangent-plane to a given surface of revolution parallel to a given plane.*

Draw a perpendicular to the given plane and a normal parallel to this line. The required tangent-plane may then be determined at right angles to the normal through the point in which it meets the surface.

The Hyperboloid of Revolution of one Sheet.

This surface is a particular case of a surface of greater generality known as the Hyperboloid of one Sheet. It is an undevelopable Ruled surface admitting of several modes

of generation, one of which is, by the revolution of an hyperbola about its conjugate axis, whence the name.

It will be treated here as a Twisted Surface of Revolution, generated by a straight line revolving about an axis not in the same plane with it; and the conjugate hyperbolas, which form the meridian section of the surface, will be determined as a type of the method by which any line whatever, given as a generatrix, may be reduced to an equivalent generator in a meridian plane.

1. *A straight line inclined at* 50° *to the horizontal plane is the generatrix of a surface of revolution, the axis of which is vertical. The nearest point in the generatrix is* ·5 *in. from the axis. Draw the elevation of a meridian section on a parallel vertical plane.*

Since every point in the generatrix describes a circle about the axis in a plane perpendicular to it, if a plane be taken through any point, P, in the generatrix perpendicular to the axis, meeting the latter in point A, and intersecting the assumed meridian plane in a straight line through A, the distance AP can be cut off on this line from A, both sides of the axis, for two points, P^1 and P^2, in the required meridian section. Similarly other points may be found.

To draw the problem, begin by describing a circle of ·5 in. radius in the horizontal plane for the plan of the *throat*, or *collar*, of the hyperboloid, and show the straight line, which is its elevation, two or three inches above xy.

Draw the given generatrix parallel to the vertical plane. In plan this will be a straight line touching the circle and parallel to xy. In elevation it will be a straight line drawn through the middle point of the elevation of the collar, and making an angle of 50° with xy.

Let o be the centre of the circle which is the plan of the collar, then o is also the plan of the axis. Determine the projections $p p'$ of any point P in the generatrix. Assume a horizontal plane passing through P to cut the surface; the plan of the section will be a circle of centre o and radius op; the elevation, a straight line parallel to xy through p'. The extremities of this elevation are two points in the required elevation of the meridian section. Others may be found in a similar manner, and the curve put in by hand.

Observations. The student should note well the following facts:—

(1) The plan of the generatrix is, in every position, a tangent to the collar.

(2) The surface has two systems of generatrices inclined at equal angles in opposite directions. Hence two straight lines, coinciding with the surface throughout their whole length, may be drawn through any point on it, and, therefore, two generatrices of different systems are always in the same plane.

(3) No two generatrices belonging to the same system are in one plane, and no three can be parallel to the same plane. Hence the surface cannot be developed.

(4) If any two generatrices of the same system be determined, and a third belonging to the other system be drawn from a point in one of them and produced, it will meet the other. Similarly, if three such generatrices be determined, and a fourth belonging to another system be drawn through any point in one of them, it will, if produced, meet the other two. Hence, we see that if three generatrices

of the surface be assumed as directrices, it might be generated by a fourth straight line which moves so as always to pass through them. That is, the Hyperboloid of Revolution is a particular case of a surface called the Hyperboloid of One Sheet, which may be defined as the surface generated by a straight line which moves so as always to meet three straight lines not in the same plane.

(5) The locus of all lines drawn through the centre of the collar parallel to the generatrices is a right conic surface called the asymptotic cone. A section plane passing through the vertex of this cone and not at right angles to the axis, cuts the Hyperboloid in an ellipse, parabola, or hyperbola, according as it meets the cone in a point, a straight line, or two straight lines. Parallel sections are similar curves.

NOTE. If the hyperbolas which form the meridian section of this surface were rotated about the transverse axis, two distinct surfaces would be produced—the Hyperboloid of Revolution of Two Sheets. This however is not a Ruled surface.

2. *Given one projection* (p *or* p′) *of a point*, P, *on the surface of the hyperboloid of revolution in the last problem, to find the other projection.*

If p be given, draw a tangent to the plan of the collar through p. This will be the plan of a generatrix. Find its elevation and the point p' in it.

If p' be given, a horizontal section of the surface through P, as in the parallel problem on the sphere, will determine p.

3. *To draw a tangent-plane to the surface at a point*, P, *on it.*

Draw the two generatrices through *P*. The plane containing these will be the plane required.

NOTE. This is a tangent-plane *at* a point only, and not *along* a generatrix.

UNDEVELOPABLE RULED SURFACES GENERALLY.

The Hyperboloid of One Sheet has been already defined as the surface produced by a straight line which moves so as always to meet three given straight lines not in the same plane.

To determine the surface, then, it is only necessary to draw a number of straight lines from different points in one of the given rectilineal directrices to meet the other two. (Prob. XII., Cor. 2, Chap. III.)

THEOREM. If any three *generatrices* be found by this construction, a straight line which moves so as always to pass through these three lines will generate the same surface.

That is, the Hyperboloid of One Sheet has two distinct systems of rectilineal generatrices, a fact which was pointed out when treating of the particular case of the Ruled Hyperboloid of Revolution.

Every plane section of the surface is a conic, and it is by a comparison of the character of the sections made by planes cutting the surface in particular directions that the student will best get a mental picture of its form. Turning to the Hyperboloid of Revolution of One Sheet, it is seen that the sections by planes at right angles to the axis are circles. Now, if these sections were ellipses, the surface would be the general case of the Hyperboloid of One Sheet we are here considering, and, from this point of view, might be called the

Elliptic-Hyperboloid. Several other modes of generation are obvious, such as by the motion of a variable ellipse, or a variable hyperbola, under particular restrictions, and by the motion of a straight line, always touching the elliptic collar, in a plane perpendicular to the plane of the collar and inclined at a constant angle to the same plane.

A number of interesting problems might be discussed on surfaces generated when the three directrices are not rectilineal, but the limits of this book exclude them.

Another group of surfaces, generated by a straight line which moves parallel to a given plane and always passes through two given lines not in one plane, may just be mentioned.

When both the directrices are straight lines, a surface called the *Hyperbolic-Paraboloid* is produced.

Like the Hyperboloid of One Sheet, this surface can be generated by the motion of a straight line in two ways. Thus, the surface may be generated by a straight line moving along two generatrices for new directrices, and always parallel to the plane to which the two original directrices are parallel.

To picture the form of this surface in the mind, the student should turn once more to the Hyperboloid of Revolution of One Sheet. Let P be a point on the collar, and assume that the meridian section through P is a *parabola*, instead of an hyperbola, of which P is the vertex. If now the section by the plane of the collar be also a parabola, with P for its vertex, instead of a circle, and the first parabola move with its vertex along the curve of the second in a plane always parallel to itself, it will generate the surface in question.

Since through every point in these two surfaces two rectilineal generatrices can be drawn a tangent-plane at a point on either surface is easily determinable.

When one directrix is a straight line and the other a curve, the surface generated is called a *Conoid*.

The only case that can be touched upon here is that in which the axis of a right cylinder and a helix traced on its surface are the directrices, and the plane of the circular base that to which the generatrix is always parallel. This is the well-known surface of the

Square-Threaded Screw.

1. *Draw two helices, ·5 in. apart and 1·5 in. pitch, on the surface of a right cylinder 2 in. diameter and vertical axis. A rectangle, ·5 in. by ·75 in., moves between these helices in a vertical plane which always passes through the axis, thus generating a square-threaded screw. Show several turns of the thread and a tangent-plane at a point on the screw surface.*

Noting that the two remote angles of the generating rectangle trace helices of the same pitch, and same distance apart, as those given, on the surface of a right cylinder of radius 1·75 in. having the same axis as the given cylinder, these helices may be drawn as in Prob. 7 on "The Cylinder" and the drawing completed to show the screw and a sectional elevation of a nut with the same thread.

To work the tangent-plane: Let P be the point taken on the helix, through P draw the rectilineal generatrix and a tangent line to the helix at that point. The plane containing the generatrix and tangent line is the required tangent-plane to the surface at P.

Before dismissing the subject of curved surfaces, there is one other Ruled Surface of Revolution of importance in the arts which may be left to the ingenuity of the student to work out for himself.

2. *An angular-threaded screw is generated by an isosceles triangle whose vertex is constrained to move in a helix while its base moves along the axis of the right cylinder on which the helix is traced. Show the screw and nut, and determine a tangent-plane to the surface at a given point on it.*

Two examples of this surface will be found in Bradley's *Elements of Geometrical Drawing*, Part II. Plate XXXVII. Figs. 1 and 2.

The student who wishes for other exercises can determine tapering, square, and V-threaded screws on a cone.

VIII.

SECTIONS BY OBLIQUE PLANES.

Preliminary.

An Oblique Plane may be given in various ways of which the following are some of the most important:—

1. By the angles α^0 and β^0 which the vertical and horizontal traces respectively make with xy. Fig. 29.

2. By the angles θ^0 and ϕ^0 which the plane makes with the horizontal and vertical planes of projection respectively. Prob. VII. Chap. III.

3. By the angles α^0 and θ^0.

4. By the angles α^0 and ϕ^0.

5. By the angles β^0 and θ^0.

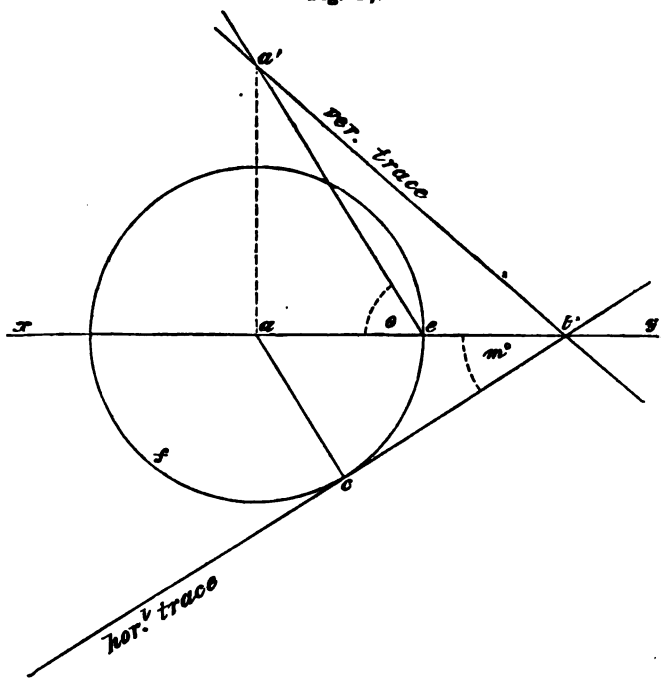
6. By the angles β^0 and ϕ^0.

7. By three given points, or their equivalents, as; (1) Two straight lines parallel or meeting, and (2) One straight line and a point. *Prob. I.* (3), *Cor. I. Chap. III.*, p. 46.

The student should solve each of these cases before going on with the work of this Chapter.

As an example, 3 and 5 (which are also types of 4 and 6) are solved in Fig. 29.

Fig. 29.



A point A (aa') in the vertical plane is made the vertex of a right cone standing on its base, *cef*, in H.P., with its generatrices inclined at θ^0. For 3, draw *ct* tangent to *cef* at *c*, meeting *xy* at *t*, and making the angle *xtc* equal to α^0; join *a't*. Then *ct* is the horizontal trace, and *ta'* the vertical trace, of the required plane. For 5, draw *a't* making the angle *a'tx* equal to β^0, and from *t* draw *tc* touching the circle *cef*. This determines the plane as before.

Nos. 4 and 6 are worked similarly, the axis of the cone, however, being in the H.P. instead of the V.P.

General Remarks on the Method of Working Sections by Oblique Planes.

When the section is of a form bounded by plane faces, the general solution is effected by finding the intersection of the given section plane with the indefinite planes of each of the faces. The parts of the indefinite intersections that are common to the faces and the section plane, are lines of the section required. Sometimes the work is simplified by finding the points in which the edges of the form pierce the section plane, by the aid of auxiliary planes containing these edges.

When the section is of a curved surface, the general method is to assume some mode of generation of the surface and construct the intersection of each generatrix with the plane.

Here also recourse is frequently had to auxiliary planes cutting the surface in generatrices and the section plane in straight lines. Sometimes, however, the auxiliary planes are made to cut the surface in some easily determinable curve other than a generatrix.

When convenient, auxiliary planes should be taken parallel or perpendicular to one of the planes of projection, as in these cases one projection of the auxiliary section is always a straight line.

For the most part the practical solution of the problems in this chapter is greatly facilitated by assuming an auxiliary plane of projection at right angles to one of the traces of the section plane. The projection of the section on this auxiliary plane being a straight line, the other projections, true shape of section, and projection of frustum on the

plane of section, may be easily deduced. The simplest practical solution of a problem is, however, not necessarily the one most meet for the purposes of the student, whose aim should always be to develop a grasp of the method and principles of the subject, and a knowledge of the properties of Form on which they are based.

PROBLEMS.

1. *A right prism standing on a square base of 2 in. side, one side of the square making an angle of 30° with* xy, *is cut by an oblique plane, inclined at 50°, which passes through a point in the axis of the prism 2 in. above the base, and has for its horizontal trace a line inclined to* xy *at an angle of 40°. Show the projections of the frustum, the true shape of the section, the development, and a new projection of the frustum on the plane of section.*

Determine plan and elevation of the prism and the traces of the section plane.

To work the projection of the section.

First Method:—Assume auxiliary vertical planes containing the edges of the prism and cutting the given plane. The lines in which the assumed vertical planes cut the given section plane, will meet the edges through which the auxiliary vertical planes are taken in required points of the section.

If the horizontal traces of these auxiliary vertical planes be taken parallel to the horizontal trace of the given oblique section plane, the lines of intersection will be horizontal—their plans coinciding with the horizontal traces of the auxiliary planes, and their elevations being lines, parallel to *xy*, drawn from the points in which the vertical traces of these planes cut the V.T. of the given section plane.

The auxiliary planes may however be taken so as to contain *two* edges and thus find two points in the section simultaneously, a particular case of which is the following:—

Second Method:—Determine the intersection of the plane of each of the faces with the given plane of section. The elevations of these intersections will cut the elevations of the edges of the prism in the required points of section, and the segments of the lines of intersection between the points of section will be the sides of the section by the given plane.

Third Method:—Take an auxiliary vertical plane of projection at right angles to the horizontal trace of the given plane of section. The projection of the section on this plane will lie in the vertical trace of the section plane on the same plane. Hence, the heights may be determined in this auxiliary elevation and transferred.

To find the true shape of the section by rabatting the plane of section into one of the planes of projection.

Let P be a point in the section of which p, p' are the projections. Through the plan, p, draw pm perpendicular to the horizontal trace and meeting it in m. Measure the hypotenuse of the right-angled triangle which has pm for its base, and pP, the height of P, for its perpendicular, along mp produced. This will give the point P rabatted into the horizontal plane about the H.T. of the section plane.

If the other points of the section be similarly rabatted, they may be joined up for the true shape of the section.

NOTE. If the auxiliary vertical plane of projection, given in the third method above, be used, the hypotenuses of these right-angled triangles may be measured at once from the auxiliary vertical trace. For this, and the development of the frustum, refer to similar work in Chap. I.

To draw the projection of the frustum on the plane of section.

Let P, Q, R, S, be the points in which the vertical edges of the prism, drawn from the points A, B, C, D, of the base respectively, meet the section plane, *i.e.* the points of the section.

The feet T, U, V, W, of the perpendiculars to the plane of section from the points A, B, C, D, will determine the projection of the base of the frustum on the plane. Lines joining the feet of the perpendiculars with the corresponding points of the section and with one another will complete the projection of the frustum on the plane of section, and the latter must now be rabatted about one of its traces to show the projection.

For example, to rabat the plane about its vertical trace, draw from p', and other points q', r', s', of the elevation of the section, perpendiculars to that trace, and mark off from it, on these perpendiculars, the actual distances of the points P, Q, R, S, from the vertical trace, *i.e.* the hypotenuses of the right-angled triangles of which perpendiculars from p', q', r', s', to the vertical trace, and from p, q, r, s, to xy, are the sides.

Join these in order for the rabatted section.

Then proceed in the same manner for the rabatment of the points T, U, V, W, which determine the projection of the base A, B, C, D, upon the plane. The rabatted points T, U, V, W, may then be joined with P, Q, R, S, respectively, for the complete projection of the frustum.

NOTE. When one point, as T, in the rabatted projection of the base upon the plane of section is found, work may be saved by joining TP, and drawing the projections of the other edges from the points Q, R, S, parallel to TP to meet

the perpendiculars from b', c', d', to the vertical trace in the points U, V, W.

Another Method :—If the auxiliary vertical trace, in the third method of working the section, be treated as a new xy, the plan of the frustum on the plane of section can be deduced without any difficulty. The student should work the problem both ways.

2. *A solid is formed by truncating a right prism, which stands upon a regular pentagonal base of* 1·5 *in. side with the nearest face of the prism parallel to the vertical plane, by two oblique planes inclined in opposite directions on either side; their horizontal traces equally inclined to* xy *at angles of* 40°, *equally distant* 1·75 *in. from the centre of the plan of the base, and the line in which they intersect inclined at* 60° *and in the same plane with the axis of the prism.*

Draw the plan of the prism and from the centre describe a circle of 1·75 in. radius. Two tangents to this circle making angles of 40° with xy will be the horizontal traces of the oblique planes; and a line from the point in which these horizontal traces intersect, through the axis and inclined at the given angle 60°, will be the common section of the two given planes, the vertical traces of which can be drawn through the vertical trace of this line.

The rest of the work is the same as in the last problem.

Develop the form and determine its projection on one of the planes of section. Work also a *new plan*, the xy for which is to be taken at 30° with the elevation of the axis.

3. *A square,* 2 *in. side, in the horizontal plane, one side inclined to* xy *at* 30°, *is the base of an oblique prism, the long edges of which are inclined at* 60° *and in plan parallel to*

a diagonal of the square. Distance between the planes of the bases 3·5 *in. Determine the section made by an oblique plane inclined in the same direction as the edges of the prism at an angle of* 30° *and passing through a point assumed in the upper base; the horizontal trace to make an angle of* 50° *with* xy.

Draw the projection of the prism and make the assumed point in the upper base the vertex of a right cone with generatrix inclined at 30°. A tangent line to the circular base of this cone and making an angle of 50° with *xy* can be drawn for the horizontal trace of the section plane. Find the vertical trace, and work the section as in the other problems.

NOTE. The method of working by means of the auxiliary vertical plane of projection will be found most useful.

Show true shape of section and plan of frustum on plane of section, and work the development.

4. *A right cylinder* 2 *in. diameter, axis* 4 *in., stands on its base and is cut by an oblique plane passing through the middle point of the axis and inclined at angles of* 60° *and* 50° *to the vertical and horizontal planes respectively. Show projections and true shape of section; also the projection of the frustum on the plane of section and its development.*

Find the oblique plane by Probs. 7 and 16, Chap. III.

Take any convenient number of generatrices of the cylinder, and find the points in which they meet the section plane, as was done in the case of the edges of the prism, Prob. 1, and join them in the elevation by a curve, which in this case will be an ellipse. In plan the base of the cylinder and the projection of the section will coincide.

For the projection of the frustum use the auxiliary vertical plane of projection as before, and develop as in Chap. I.

NOTE. The plan of the section of any right prism standing on its base, or cylindrical surface with generatrices perpendicular to the horizontal plane, by an inclined or oblique plane, coincides with the trace of the form on the horizontal plane of projection. Hence, any given figure whatever may be the plan of a certain figure lying in a given oblique plane: and the elevation and true shape of this figure may be deduced in the same way as a section of a vertical prism, or cylinder, having the given figure for its horizontal trace, would be worked if the given plane were the plane of section.

5. *To work the section of an oblique cylindrical surface by an oblique plane.*

In the particular case in which the horizontal trace of the cylinder is a circle, or part of a circle, a series of auxiliary horizontal planes cutting the surface in similar circles, and the section plane in lines parallel to its horizontal trace, would be most convenient.

In the general case, in which the horizontal trace of the cylinder is any curved line whatever, draw a number of convenient generatrices and determine the points in which these generatrices meet the section plane. An auxiliary vertical plane of projection at right angles to the horizontal trace of the section plane facilitates finding the required points in the section.

6. *Determine a regular hexagonal pyramid of* 1·25 *in. side standing on its base, axis* 4 *in. long, and one side of the hexagon making an angle of* 20° *with* xy. *Section by an*

oblique plane passing through a point in the axis 1·5 ***in. above the base; the horizontal and vertical traces making angles of*** 80° ***and*** 40° ***respectively with*** xy.

A line from the given point in which the section plane meets the axis, parallel to H.P. and inclined at 80° to the V.P., will determine a point in the vertical trace of the section plane from which the traces of the plane may be at once found.

Determine the section (1) as in Prob. 1, by the aid of auxiliary vertical planes containing the edges of the pyramid; (2) by finding the intersections of the given oblique plane with the indefinite planes of the faces of the solid; or (3) by means of an auxiliary elevation on a vertical plane taken at right angles to the horizontal trace of the oblique plane of section.

When the problem is worked by means of auxiliary vertical planes containing the slant edges, the work may be shortened if it be noted that the point in which the axis of the pyramid meets the section is common to all the intersections. It will be, therefore, only necessary to find a second point in each intersection—such as that in which the vertical or horizontal trace of any one of these vertical planes cuts the corresponding trace of the section plane—to determine the line of intersection and, therefore, another point in the required section.

Find the true shape of the section by rabatting the section plane as in other problems.

To draw the projection of the frustum on the plane of section.

(1) Work for the points in the section as in Problem 1.

(2) Treat the projection of the vertex on the plane of section as the points *T*, *U*, &c. were treated in the same problem.

(3) The lines through the rabatted projection of the vertex on the section plane and the points of section just determined, will be the indefinite projections of the slant edges of the pyramid. The points of the base may be fixed by perpendiculars from their elevations to the vertical trace to intersect these indefinite projections of the edges, and when found may be joined consecutively to complete the required projection of the frustum.

The *development* may be found by describing a circle with radius equal to a slant edge of the pyramid, setting along the circumference the edges of the base as chords, on one of which the square for the base must be described, and joining these points to the centre of the circle. The points of section may be found by dividing the developed edges in the *same proportion* as the corresponding plans or elevations are divided by the projections of the points of section. Fit on the true shape of the section.

7. *Assume an irregular hexagonal pyramid and cut it as in the last problem by an oblique plane. Show the true shape of the section, projection of frustum on plane of section, and development.*

To work this problem, use an auxiliary vertical plane of projection at right angles to the horizontal trace of the given oblique plane.

8. *A right cone, axis* 3·5 *in., diameter of base* 2 *in., stands upon its base and is cut by an oblique plane which passes through three given points* X, Y, Z, *on its surface. Show true shape of section, projection of frustum on plane of section, and development.*

Find the traces of the plane which contains the three given points.

To work the section assume a series of horizontal auxiliary planes cutting the cone and the plane of section. The sections of the cone will be circles, and of the plane straight lines, parallel to the horizontal trace, drawn from the points in which the vertical trace of the section plane meets the vertical traces of the auxiliary planes. The points common to the circles and the lines of section will be required points in the section of the cone by the given plane.

Complete as in other problems. For the development see Chap. I.

9. *To work the section by a given oblique plane of a conical surface whose directrix is any curve whatever and vertex a point anywhere in space.*

Find a convenient number of generatrices and determine their intersections with the given section plane. The line drawn through these points will be the section required.

10. *A right hexagonal prism,* 1·25 *in. side, axis* 4 *in. long, horizontal, and inclined at* 35° *to V. P.; lowest edge* ·75 *in. above H. P., and a face containing it inclined at* 20°, *is cut by an oblique plane which passes through three points to be assumed at pleasure on any three edges of the solid. True shape of section, projection of frustum, &c. as in other problems.*

Use either vertical or horizontal auxiliary planes containing the edges.

11. *Take a right cylinder, axis horizontal, inclined at* 35° *to V. P., instead of the prism in the last problem, and work oblique section, true shape, &c.*

(2) Treat the projection of the vertex on the plane of section as the points *T*, *U*, &c. were treated in the same problem.

(3) The lines through the rabatted projection of the vertex on the section plane and the points of section just determined, will be the indefinite projections of the slant edges of the pyramid. The points of the base may be fixed by perpendiculars from their elevations to the vertical trace to intersect these indefinite projections of the edges, and when found may be joined consecutively to complete the required projection of the frustum.

The *development* may be found by describing a circle with radius equal to a slant edge of the pyramid, setting along the circumference the edges of the base as chords, on one of which the square for the base must be described, and joining these points to the centre of the circle. The points of section may be found by dividing the developed edges in the *same proportion* as the corresponding plans or elevations are divided by the projections of the points of section. Fit on the true shape of the section.

7. *Assume an irregular hexagonal pyramid and cut it as in the last problem by an oblique plane. Show the true shape of the section, projection of frustum on plane of section, and development.*

To work this problem, use an auxiliary vertical plane of projection at right angles to the horizontal trace of the given oblique plane.

8. *A right cone, axis* 3·5 *in., diameter of base* 2 *in., stands upon its base and is cut by an oblique plane which passes through three given points* X, Y, Z, *on its surface. Show true shape of section, projection of frustum on plane of section, and development.*

Find the traces of the plane which contains the three given points.

To work the section assume a series of horizontal auxiliary planes cutting the cone and the plane of section. The sections of the cone will be circles, and of the plane straight lines, parallel to the horizontal trace, drawn from the points in which the vertical trace of the section plane meets the vertical traces of the auxiliary planes. The points common to the circles and the lines of section will be required points in the section of the cone by the given plane.

Complete as in other problems. For the development see Chap. I.

9. *To work the section by a given oblique plane of a conical surface whose directrix is any curve whatever and vertex a point anywhere in space.*

Find a convenient number of generatrices and determine their intersections with the given section plane. The line drawn through these points will be the section required.

10. *A right hexagonal prism,* 1·25 *in. side, axis* 4 *in. long, horizontal, and inclined at* 35° *to V. P.; lowest edge* ·75 *in. above H. P., and a face containing it inclined at* 20°, *is cut by an oblique plane which passes through three points to be assumed at pleasure on any three edges of the solid. True shape of section, projection of frustum, &c. as in other problems.*

Use either vertical or horizontal auxiliary planes containing the edges.

11. *Take a right cylinder, axis horizontal, inclined at* 35° *to V. P., instead of the prism in the last problem, and work oblique section, true shape, &c.*

Use a series of horizontal auxiliary planes cutting the cylinder in generatrices.

12. *Take a right pentagonal pyramid, side of base* 1·5 *in., axis* 3·5 *in., when its highest face is horizontal and the plan of its axis inclined at* 40° *to* xy, *and work a section of it by an oblique plane.*

Use auxiliary vertical planes containing the edges, except for the edges of the upper face, in which case a horizontal auxiliary plane will be more convenient.

NOTE. Since all the vertical planes containing the long edges of the pyramid must also contain a perpendicular to the horizontal plane from the vertex of the solid, all the intersections of these planes with the plane of section must pass through the point in which the perpendicular in question meets this section plane. Hence this point may be used to simplify the work, just as the intersection of the axis was used in Problem 6.

13. *Work section by an oblique plane of a right cone with its axis horizontal and inclined to the V. P. at* 40°.

Use a series of auxiliary vertical planes cutting the cones in generatrices, and observe that the "Note" to the last problem applies also to this. Each plane will, in general, give two points.

14. *Work the section of a sphere by an oblique plane which passes through two given points on its surface and makes an angle of* 60° *with H. P.*

Determine the plane, and for the section use auxiliary planes parallel to one of the planes of projection.

15. *Section of the Hyperboloid of Revolution of one Sheet, given in the last Chapter, by an oblique plane.*

Use horizontal auxiliary planes. These will cut the surface of revolution in circles and the plane of section in straight lines—the work being similar in every respect to that of Problem 8 above.

Any surface of revolution with its axis at right angles to one of the planes of projection might be worked in the same way; *e.g.* the *sphere* in Prob. 14 above, the *spheroid* Prob. 1, "Surfaces of Revolution," Chap. VII., and the following :—

16. *Section of the annulus or anchor ring, Prob.* 5, *"Surfaces of Revolution," Chapter VII., by a tangent-plane which passes through its centre and makes an angle of* 60° *with the vertical plane of projection.*

17. *Section by an oblique plane of the screw surfaces given at the end of Chap. VII.*

Use auxiliary elevations at right angles to the H. T. of the section plane, and deduce therefrom the required projections of the points in which the generatrices meet that plane.

18. *To find the section by an oblique plane of a surface of revolution with its axis oblique to both planes of projection and not at right angles to the oblique plane.*

Use auxiliary planes of section perpendicular to the axis of the surface of revolution. These cut the oblique plane in straight lines and the surface in circles. The points common to these lines and circles are points in the curve of intersection.

NOTE. The difficulty of projecting the surface of revolution in the position given is generally sufficient to deter

any but the most courageous of students from attempting this problem. It may, however, be remarked that the projection of a surface of revolution on a plane inclined to its axis, is the section by that plane of a cylinder enveloping the surface and perpendicular to the plane of projection. For the construction of a cylinder enveloping a surface of revolution see Prob. 4, "Problems on Surfaces of Revolution," Chap. VII.

IX.

INTERSECTIONS.

General Remarks.

THE methods by which the intersection of two surfaces are determined when one of them is a plane, have been pretty fully exemplified in the preceding chapter, and it remains now only to comment briefly on the general case.

If a section plane be taken through any two intersecting surfaces whatever, and the lines of section with each of the surfaces be drawn, the point, or points, in which the lines of section meet each other, will be in the intersection of the surfaces. Consequently, by taking a *series* of these auxiliary section planes, a sufficient number of points may be found, and the line of intersection drawn through them.

Although the auxiliary planes may be chosen arbitrarily, it is obviously convenient, in general, to take them parallel, or perpendicular, to one of the planes of projection, so that one projection of the section shall always be a straight line. This rule has, however, sometimes to give way to other considerations, as when, for instance, we are dealing with Ruled Surfaces, in which case it often facilitates the work if the auxiliary planes be so taken as to cut the surface along its rectilineal generatrices.

NOTE. For the application of the principle of parallel sections to the intersection of forms given by their *figured plans*, see Prob. 11, Chapter X., "Figured Projections and Scales of Slope."

FORMS BOUNDED BY PLANE SURFACES.

1. *Two right prisms, with square bases of 2·5 in. side, intersect. One is placed with its edges horizontal, and parallel to the vertical plane; its lowest face inclined 30°, and its lowest edge 1 in. above the horizontal plane. The other stands on its base with one face inclined 25° to the vertical plane, and its axis passing through the middle point of the highest edge of the horizontal prism. Axes of the prisms 4 in. long. Draw plan and elevation showing the intersection, and develop the vertical prism.*

To get the prisms into position:—

Commence with an auxiliary elevation of the horizontal prism on a vertical plane perpendicular to its axis, so that the elevations of both bases coincide. The elevation of the prism in this position will be a square, 2·5 in. side, the lowest corner 1 in. above, and one side making an angle of 30° with *xy*. For the final elevation take a new *xy* parallel to the plan of the axis.

For the plan of the vertical prism, describe a circle of 2·5 in. diameter from the middle point of the plan of the highest edge of the horizontal prism as centre. A pair of parallel tangents to this circle making angles of 25° with the new *xy*, and a second pair at right angles to the former, will determine the square base of the vertical prism in the required position. Thence determine the elevation on the new vertical plane.

To find the intersection :—

The forms being bounded by plane faces, the lines of intersection will be straight. And since but one straight line can be drawn between two points, it is only necessary to determine two points in each of the intersections of the faces, and to draw the line of intersection through them. The points most convenient for the purpose are those in which each of the edges of the one prism pierces the faces of the other.

Begin with the edges of the horizontal prism. The plans of the points in which these meet the faces of the vertical prism are shown where the plans of the horizontal edges of the one prism cut the square which is the plan of the other. Determine the elevations of these points by drawing projecting-lines from their plans to meet the elevations of the edges in which they lie.

Observe :— *Each edge should be considered separately, and as much of it as is visible be put in as a dark continuous line; any part that is invisible should be dotted, and that portion which passes through the other prism rubbed entirely out.*

Unless this rule be strictly attended to before any attempt is made to join up for the intersection, inextricable confusion will in general be the end of an abortive effort the reverse of instructive.

For the edges of the vertical prism, Measure the heights at which they pierce the faces of the horizontal prism from the auxiliary elevation, and transfer them to the new vertical projection.

Treat each edge of the vertical prism in the same way as recommended for the horizontal edges in the observation printed in italics above.

The lines of intersection can, at this stage, be easily put in from inspection.

To work the development:—

Develop the entire prism as in Prob. 3, Chap. 1, and for the development of the intersection, find, first, the *points on the edges*, by measuring from one of the elevations, and, second, the *points in the faces*. These are determined by their distances from the edges and their heights above the base.

Prick off the development on a sheet of cardboard and make the model of the intersection.

2. *Using the same plan and auxiliary elevation as in the preceding problem, make a new elevation on a vertical plane inclined at* 40° *to the long edges of the horizontal prism, so as to show the parts which were invisible in the elevation in Problem* 1. *Develop the horizontal prism and deduce a second plan of the intersecting solids*—xy *at pleasure.*

3. *A regular pentagonal prism,* 1·5 *in. side,* 5 *in. long, stands on the horizontal plane with the vertical face passing through the side of the base nearest to* xy *parallel to V.P., and is intersected by a similar and equal prism with its axis horizontal,* 2·5 *in. high, inclined at* 30° *to V.P. and* ·25 *in. from the axis of the other. The rectangular face of the horizontal prism that is furthest from V.P. to be vertical. Plan, elevation, and development.*

Begin with a plan of the vertical prism at a convenient distance in front of *xy*.

Since the axis of the horizontal prism is inclined at 30° to V.P. the base must be inclined at 60°, the complement of the angle of inclination of the axis, to the same plane.

Determine an auxiliary vertical plane of projection parallel to the base of the horizontal prism, and show thereon an elevation of the vertical prism. A point in this elevation, 2·5 in. above *xy* and ·25 in. from the elevation of the axis of the vertical prism, will be the elevation of the axis of the horizontal one, and if a circle be described from this point, with the proper radius, the pentagon which is the auxiliary elevation of the horizontal prism may be put about it with one side vertical, and the plan and new elevation be deduced without difficulty.

The intersection is worked as in Probs. 1 and 2.

4. *A right prism, 4 in. long, bases equilateral triangles of 2 in. side; axis horizontal, inclined 30° to V.P.; lowest edge ·25 in. above, and a face containing it inclined 15° to H.P., intersects a right pyramid, square base 2·5 in. side, axis 3 in. long, vertical, and passing through the middle point of the axis of the prism. Side of square base inclined at 30° to* xy *and not parallel to the long edges of the prism. Show intersection in plan and elevation and work development of the pyramid.*

Determine plan and elevation of prism as in Problem 6, Chap. I., page 31, by means of an auxiliary elevation on a plane at right angles to the axis.

For the pyramid;— From the middle of the plan of the axis of the prism as centre, describe a circle of 2·5 in. diameter and circumscribe it by a square, in the right position, for the plan of the base of the pyramid. Thence determine its elevation, and also an elevation on the auxiliary vertical plane of projection which was used to find the plan of the prism.

To work the intersection:—

(1) Find the projections of the points in which the edges of the pyramid pierce the faces of the prism.

The elevations of these points are shown in the auxiliary plane where the elevations of the edges of the pyramid cut the equilateral triangle which is the projection of the prism on that plane. From these the other required projections may be found by drawing projecting-lines from them to meet the other projections of the lines that contain the points.

When these points are found, put in the edges of the pyramid in the way recommended for the edges of the prism Prob. 1.

(2) Find the projections of the points in which the edges of the horizontal prism pierce the pyramid:—

Assume that the three corners of the equilateral triangle in the auxiliary elevation are the elevations of the points in which the horizontal edges of the prism meet the faces of the pyramid, and, on this assumption, draw lines from the vertex of the pyramid through these points in its faces and produce them to the base. The projections of these lines on the auxiliary plane will be drawn from the projection of the vertex on that plane through the three corners of the triangle. Each of these is the elevation of *two* lines—one passing through the point in which the edge of the prism enters, and the other through the point in which it leaves the pyramid. Determine these lines in plan. The points common to these lines and the plans of the horizontal edges of the prism will be the plans of the required points, from which the other elevations may be at once obtained.

Put in the projections of the edges in the proper manner and then join up for the projections of the intersection.

Develop the pyramid as in Prob. 1, Chap. 1. The *points in the faces* may be readily found by showing in the develop-

ment the lines which pass through them, and determining them in these lines in the same way as the points in the edges are found.

5. *For the prism in the preceding problem substitute an irregular pentagonal prism with a re-entering angle. No face to be either horizontal or vertical: axis of pyramid to pass through the middle point of the highest edge of the prism, and the latter form to have two long edges—one on each side of the axis—entirely free from the pyramid. Other conditions as in Problem* 4. *Plan, elevation, and development of pyramid showing the lines of intersection.*

Begin with a plan of the pyramid and an auxiliary elevation on a vertical plane parallel to the plane of the base of the prism. The end elevation of the prism may then be put in so as to fulfil the conditions, and the problem worked as in Prob. 4 above.

6. *Intersection of a right square pyramid and a regular pentagonal prism.*

Pyramid:—Axis 4 *in. long, inclined at* 35° *to H.P., and in plan making an angle of* 25° *with* xy: *side of square base* 2 *in. and inclined at* 30° *to the horizontal trace of the plane of the base: lowest corner resting upon H.P.*

Prism:—Standing on the H.P. with a regular pentagon of 1·5 *in. side for its base; axis* 3·5 *in. long and vertical.*

Intersection when the prism is so placed that the vertical face furthest from xy *is parallel to the V.P. of projection, and has one of its long edges free from the pyramid. All other edges of the prism to pass through the pyramid and one of them to meet it at a point in its base.*

Work also a development of the pyramid.

The points in which the edges of the pyramid meet

the vertical faces of the prism are shown in plan where the plans of these edges cross the sides of the pentagonal base, and can thence be determined in elevation.

NOTE :—It sometimes happens that an edge of the pyramid pierces a base of the prism, in which case the point of intersection is readily deduced from the elevation.

The intersections of the vertical edges of the prism with the faces of the pyramid can be found *either* by the general method for the intersection of a straight line and a plane, *or* by the following equivalent special construction:—

Draw lines from the plan of the vertex of the pyramid through the points which are the plans of the vertical edges of the prism, and produce them to the base of the pyramid. Assuming each of these lines to be the plan of two lines—one drawn from the vertex through the point in which the edge of the prism enters one face of the pyramid and the other from the vertex through the point in which it leaves another—show them in the elevation. The intersections of the elevations of these lines with the elevations of the edges through which they are drawn in plan, give the elevations of the required points of intersection.

The intersection of the edge of the prism with the base of the pyramid may be similarly found by drawing a line, say, from one corner of the plan of the square base, through the plan of the vertical edge meeting it, and showing this line in the elevation of that base.

Complete the representation of the *edges* of the intersecting forms and *then* join up the points of intersection from inspection.

7. *A right hexagonal pyramid,* 1·75 *in. side, axis* 5 *in.*

long, stands on its base with one side making an angle of 40° *with* xy, *and is intersected by a right square prism, 2 in. side, axis passing through that of the pyramid, inclined* 30° *to H.P., and in plan making an angle of* 50° *with* xy. *No side of the square to be horizontal. Show plan, elevation, and intersection.*

Develop the prism.

Find the points in which the edges of the pyramid meet the faces of the prism, by the aid of auxiliary vertical section planes cutting the prism and containing the slant edges of the pyramid. Similarly, find the intersections of the edges of the prism with the faces of the pyramid by the aid of vertical planes containing the former.

8. *Work the following general cases:—*

α. *Two intersecting prisms of four and six sides respectively, axes oblique to both planes of projection. Other data at pleasure.*

β. *Two intersecting pyramids of four and five sides respectively, axes oblique to both planes of projection. Other data as above.*

γ. *General case of irregular prism and pyramid.*

When the traces of the indefinite planes of the faces fall within the limits of the paper the intersections of these planes may be found by Prob. 12, Chap. III. and that of the forms put in therefrom. As, however, it seldom happens that all the traces of the planes intersect within the limits of the drawing, recourse is had to auxiliary section planes. Two such planes—one parallel to one plane of projection and the other to the other—will be found most convenient for the purpose. From the points furnished by the sections, and such of the traces as meet on the paper, the intersections can be put in without much difficulty.

FORMS BOUNDED BY CURVED SURFACES.

As far as possible the following exercises have been grouped according to the most convenient methods of working them. Some of the problems admit of several constructions, but on the whole it will be found that the problems worked in each group afford types of the most approved method of working those of the same group to which no solutions are appended.

"*Critical lines*" *and* "*Key points.*" In all intersections that involve curved surfaces the student should direct his attention to the lines in which the projecting surfaces touch the curved forms and the points in which these lines on the one form meet the surface, or surfaces, bounding the other.

The line of intersection always passes through these points, and inasmuch as a habit of attending to them, and to their corresponding lines, greatly simplifies matters by developing a power of prospective insight into the characteristic features of the intersection resulting from any given combination, they have been specially designated the *critical lines* and *key points.*

GROUP I.

1. *Intersection of a right cone and a sphere. Cone standing on its circular base of* 2·5 *in. radius, axis* 3 *in. long: sphere*

resting upon H.P., radius** 1·25 **in., centre** 1 **in. from the axis of the cone; line joining the plans of the centre of the sphere and the vertex of the cone inclined at** 30° **to** xy. ***Development of cone showing curve of intersection.

Take a series of sections of both surfaces by auxiliary planes parallel to H.P. The elevation of each will be a straight line parallel to *xy*, and the plan, two circles—one described from the plan of the centre of the sphere and the other from the plan of the vertex of the cone, as centres, with radii that may be measured from the elevation. The points in which each pair of circles intersect will be points in the plan of the intersection of the forms, and from these the elevations may be found by projecting, and the projections of the intersection put in.

NOTE. It will simplify the work if the auxiliary planes are taken at equal distances above and below the centre of the sphere—so that one circle in plan shall serve for two sections of that surface.

For the development.

(1) Divide the base of the cone in plan into a number of equal parts and put in the plans of the generatrices from the points of division to the vertex.

(2) Develop the cone and show these generatrices in the development.

(3) Mark the points where the curve of intersection, *in plan*, meets the plans of the generatrices, and divide the corresponding generatrices in the development in the same proportion, for the points through which the curve of intersection passes.

2. *Arrange an example of two intersecting spheres of*

unequal sizes and unequal heights; the line joining their centres oblique to both planes of projection.

3. *Intersection of a sphere and a surface of revolution with vertical axis*—e.g. *the Spheroid, or the Hyperboloid, in Chap. VII.*

4. *Intersection of a right cone and a right cylinder with their axes vertical.*

NOTE. The plan of the intersection in this problem, and in the next, coincides with the circle that is the plan of the cylinder.

5. *Intersection of a sphere and a right cylinder standing on its base.*

NOTE. Any surface of revolution with its axis vertical might be substituted for the sphere in this problem.

6. *Arrange an example of a sphere, or a spheroid, axis vertical, intersecting an oblique cone or cylinder with a circular horizontal trace.*

7. *Intersection of a sphere with the anchor ring, Prob.* 5, "*Surfaces of Revolution*," *Chap. VII.*

8. *Intersection of an oblique cone, or a cylinder, having a circle for its horizontal trace, with an anchor ring, axis vertical.*

9. *Intersection of a right cone standing on its base and an oblique cone, or cylinder, with circular horizontal trace.*

10. *Work one of the following combinations :—*

α. *Two oblique cylinders with circular horizontal traces.*

β. *Two oblique cones with circular horizontal traces.*

γ. *Oblique cone and oblique cylinder with circular horizontal traces.*

GROUP II.

1. *Intersection of two right cylinders. One 2 in. diameter; axis, horizontal, 2·5 in. above H.P., inclined 40° to V.P., and 5 in. long; the other 2·5 in. diameter, axis vertical and 5 in. long.*

α. *When the axis of the vertical cylinder passes through the middle point of the horizontal one.*

β. *When the axis of one is ·25 in. from that of the other so that the cylinders have a common tangent-plane.*

γ. *When the axis of one is ·5 in. from that of the other.*

Develop the horizontal cylinder in the last.

To find the intersection:—Divide one base of the horizontal cylinder into any number of equal parts (say sixteen), and through the points of division draw straight lines (*i. e.* generatrices) on the cylindrical surface, and show them in plan and elevation.

(1) *To show the lines in plan:*—Rabat the base about its horizontal diameter till parallel to the H.P. This will be a circle described upon a short side of the rectangle which is the plan of the horizontal cylinder. Draw half of the circle only and divide it into eight equal parts. Lines from the points of division, parallel to the plan of the axis, will determine the plans of the generatrices—each line, with the exception of the two outside ones, representing two generatrices—one above and one below the level of the axis.

(2) *To show the lines in elevation:*—Rabat the base about its vertical diameter till parallel to the V.P. This

will be a circle described upon the major axis of the ellipse which is the elevation of the base of the horizontal cylinder. Only half of this circle need be shown, and this, if divided into eight equal parts, will furnish points through which lines parallel to the elevation of the axis may be drawn for the elevations of the generatrices. These lines in elevation, with the exception of the highest and lowest, will each represent two generatrices—one on the front and one on the back part of the cylindrical surface.

NOTE. *Any horizontal lines whatever* may be taken on the cylinder in plan, and their elevations determined in the vertical projection by the aid of an auxiliary elevation of the horizontal cylinder on a vertical plane at right angles to its axis, whence the heights can be measured: but it is much more convenient to take them at equal distances apart, especially when a development has to be worked.

Having shown these generatrices in plan and elevation, the points in which they meet the surface of the vertical cylinder may be determined for points in the curve of intersection.

In plan, the curve of intersection coincides with the circle that is the plan of the vertical cylinder, and the plans of the points in which the lines on the horizontal cylinder meet the surface of the vertical cylinder are shown where the plans of these horizontal lines cross its circular base. Projecting-lines, drawn from these points to meet the corresponding horizontal lines in elevation, determine the points through which the elevation of the line of intersection is to be drawn.

NOTE. Pay attention to the *critical lines* and *key points*, and number the generatrices in plan and elevation to avoid confusion.

The development may be worked as in preceding chapters.

2. *Intersection of a right cylinder with a right cone. Cylinder* 2·5 *in. diameter, axis* 5 *in. long and vertical; cone with base* 2·5 *in. diameter; axis,* 5 *in. long, horizontal, inclined* 45° *to V.P. and* 2·5 *in. above H.P.*

α. *When the axis of the cylinder passes through the middle point of the axis of the cone.*

β. *When the two surfaces have a common tangent-plane.*

γ. *When part of the cone is without the cylinder.*

Develop the cone in β.

Divide the base of the cone into a number of equal parts, as was done with the base of the cylinder in the foregoing problem, and show plan and elevation of the generatrices.—The lines in this case instead of being parallel to the axis will radiate from the vertex to the points on the base, and the student must observe that although in the *plan* each line will, as in the cylinder, represent two generatrices, in the *elevation* this will not be the case, *i. e.* the front and back lines not being parallel to the axis will not coincide.

When the plans and elevations of the generatrices are determined, the intersection is found as before.

The development is worked as in other chapters.

3. *Arrange a right cylinder standing on its base and find its line of section with a right cylinder whose axis is inclined to both planes of projection.*

Take any number of convenient points in the horizontal trace of the oblique cylinder and draw plans and elevations of generatrices passing through them. The elevation of the

intersection can then be found from the points where the plans of the generatrices cross the horizontal trace of the vertical cylinder, as in the other problems of this group.

NOTE. The oblique cylinder might be assumed with a circle for its horizontal trace and worked as above, but in this case it could be as conveniently worked by a series of horizontal auxiliary sections, as in Group I.

4. *Arrange a vertical cylinder and an oblique cone and find their intersection.*

The generatrices in this case are drawn to the vertex of the cone; with this exception the description of the construction for the preceding problem, and the "NOTE," apply to this.

5. *Find the intersection of the anchor ring, Prob.* 5, *"Problems on Surfaces of Revolution," Chap. VII., with a vertical cylinder.*

Divide a vertical circular section of the ring into a number of equal parts, as for the base of the cylinder, Prob. 1, of this group, and through the points of division draw horizontal circular lines on its surface. The ring when thus prepared may be likened in some respects to a long cylinder bent into a circular form, and the intersection will be found to present no greater difficulty than that of the horizontal cylinder in Prob. 1 above.

NOTE. This same construction, looked at in another aspect, might be regarded as an illustration of the method of working by a series of parallel sections as in Group I.

GROUP III.

1. *Intersection of a right cone standing on a circular base of* 5 *in. diameter, axis vertical and* 3 *in. long, with a right*

cylinder resting upon H.P.; base 2·5 *in. diameter, axis horizontal,* 5·5 *in. long, inclined* 45° *to* xy, *and in plan* ·75 *in. distant from the plan of the vertex of the cone. Develop the cone with curve of intersection.*

Determine horizontal lines (generatrices) on the surface of the cylinder as in the horizontal cylinder, Prob. 1, Group II., and work sections of the cone by auxiliary horizontal planes containing these lines.

In elevation the sections of the cone will be represented by the segments of the elevations of the generatrices that lie between the legs of the triangle forming the elevation of the cone.

In plan these sections will be circles, described from the plan of the vertex of the cone as centre, with radii measured from their corresponding elevations.

The points in plan in which the generatrices of the cylinder meet the corresponding circles of the same level, will be points in the plan of the intersection, through which the curve must be drawn by hand.

Projecting-lines, drawn from these points to the elevations of the generatrices in which they lie, will determine points in the elevation of the curve of intersection.

2. *Intersection of a horizontal cylinder with a sphere.*

Sphere 4 *in. diameter, and centre* 2·5 *in. high.*

Cylinder 2 *in. diameter, axis horizontal, inclined* 45° *to V.P., in plan* ·5 *in. from, and in elevation* ·75 *in. below, centre of sphere.*

Develop the cylinder.

3. *Intersection of a horizontal cylinder with the Hyperboloid of Revolution of one sheet given in Chapter VII.*

4. *A surface generated by the revolution of a circle of* 3 *in. diameter about a tangent-line as axis, rests upon H.P. with the axis vertical; a sphere of* 2·5 *in. diameter rolls along the H.P. on a line inclined at* 40° *to V.P. and tangential to the circle in which the surface of revolution touches the H.P. Show the opening through which the sphere would pass.*

5. *Arrange an example of a horizontal right cylinder, axis inclined to V.P., intersecting an oblique cone, or oblique cylinder, which has a circle for its horizontal trace.*

Group IV.

1. *Arrange an example of a horizontal right cylinder, axis inclined to V.P., intersecting another right cylinder, axis oblique to both planes of projection, and show the curve of intersection.*

Auxiliary sections of the cylinders determined by horizontal planes as in Group III. would cut the horizontal cylinder in generatrices and the oblique cylinder in ellipses. The labour, however, of drawing these curves would be considerable, and it is, therefore, better to take auxiliary sections by planes containing generatrices of the horizontal cylinder and parallel to those of the oblique cylinder.

(1) Prepare the horizontal cylinder by drawing a number of rectilineal generatrices on its surface, as in preceding problems.

(2) Assume a point in a generatrix of the horizontal cylinder and draw therefrom a line parallel to the generatrices (or axis) of the oblique cylinder.

(3) Find the H.T. of this line and through it draw a straight line in the H.P. parallel to the plan of the generatrices of the horizontal cylinder.

The line last drawn will be the H.T. of an auxiliary plane containing a generatrix of the horizontal cylinder and parallel to the generatrices of the oblique cylinder.

(4) From each of the two points in which, in general, the H.T. of this auxiliary plane meets the curve which is the horizontal trace of the oblique cylinder, draw plans of the generatrices of the latter surface. These will be the plans of the lines in which the auxiliary section plane cuts the oblique cylinder, and the points in which these lines meet the generatrix of the horizontal cylinder will be points in the plan of the curve of intersection of the surfaces. Similarly, other points may be found, by taking auxiliary planes through each of the generatrices of the horizontal cylinder, and the curve put in by hand.

The elevation may be readily found from the plan.

2. *Arrange an example of a horizontal right cylinder, axis inclined to V.P., intersecting an oblique right cone, and show the curve of intersection.*

Take the auxiliary section planes in this case containing generatrices of the horizontal cylinder and the vertex of the oblique cone.

NOTE. To find the H.T. of the section planes, the lines drawn from the assumed points in the generatrices of the horizontal cylinder, in the last problem parallel to the axis of the oblique surface, must now be drawn from the vertex of the cone. The oblique cylinder may in fact be likened to a cone with its vertex at an infinite distance.

3. *Arrange a right cone standing on its base intersecting a cylindrical surface oblique to both planes of projection, and*

show curve of intersection. The H.T. of the cylinder not to be a circle.

Take auxiliary section planes containing the vertex of the cone and parallel to generatrices of the cylinder.

NOTE. The horizontal traces of these planes will all pass through the H.T. of a line drawn from the vertex of the cone parallel to the generatrices of the oblique cylinder.

4. *Arrange a right cone standing on its base intersecting another right cone whose axis is inclined at, say, 40° to the V.P., and horizontal.*

Take auxiliary planes containing generatrices of the cone whose axis is horizontal and passing through the vertex of the one whose axis is vertical.

NOTE. All the horizontal traces of these planes will pass through the H.T. of the line joining the vertices of the two cones.

(1) Divide the base of the horizontal cone into equal parts and determine the lines on its surface as in Prob. 2, Group II.

(2) Draw a straight line from the vertex of the vertical cone through any convenient point in a generatrix of the horizontal one and find its H.T. This will be a second point in the H.T. of the auxiliary plane containing the generatrix. Similarly, the horizontal traces of the other auxiliary planes may be determined, and the intersection worked as before.

5. *Arrange a right cone standing on its base and find its intersection with another right cone, axis oblique to both planes of projection.*

GENERAL CASES.

1. *To find the intersection of two cylindrical surfaces oblique to both planes of projection.*

Take auxiliary section planes parallel to the generatrices of both surfaces.

NOTE. Some little difficulty is usually experienced in getting the cylinders into a position that will give a satisfactory intersection, unless somewhat cumbrous and elaborate data are supplied for their arrangement. One of the simplest ways of arranging an intersection of these surfaces is to begin by drawing two lines inclined at an angle of about 120°, and treat these as the plans of the axes of the intersecting cylinders. Let one, for example, be a right cylinder of 2·5 in. diameter, axis inclined 55°, and the other a right cylinder of 2 in. diameter, axis inclined at 40°, and, say, a little below the other.

To find their intersection. Determine plan and elevation of any assumed point in space, and the horizontal traces of two lines drawn from the point, one parallel to the axis of the one cylinder, and the other parallel to the axis of the other. The line joining the horizontal traces of these lines will be the H.T. of a plane containing them, and therefore of a *plane parallel to the generatrices of the two surfaces.*

A series of lines parallel to this horizontal trace can then be taken, cutting the horizontal traces of the cylinders, for the horizontal traces of the auxiliary planes.

Lines from the points in which the horizontal traces of the auxiliary planes meet the horizontal traces of the surfaces, parallel to the plans of their respective axes, determine points in the plan of the curve of intersection, which may be drawn in by hand and projected for the elevation.

Attention should be given to the *critical lines* and *key points.*

2. *To find the intersection of two conical surfaces oblique to both planes of projection.*

Take auxiliary section planes passing through the vertices of both cones and, therefore, cutting both surfaces in generatrices.

NOTE. The horizontal traces of all the auxiliary section planes pass through the H.T. of the line joining the vertices of the two surfaces, and the vertical traces of the same planes pass through the V. T. of that line.

Sometimes, when the vertical trace of one of the cones is more accessible than its horizontal trace, the vertical traces of the auxiliary section planes are required.

3. *To find the intersection of an oblique cylindrical surface with an oblique cone.*

Take auxiliary section planes passing through the vertex of the cone and parallel to the generatrices of the cylinder.

NOTE. The horizontal traces of these auxiliary planes all pass through the H.T. of a line drawn from the vertex of the cone parallel to the generatrices of the cylinder.

GROUP V.

1. *Arrange an oblique right cylinder intersecting a sphere, and find the curve of intersection.*

If an auxiliary section of the two surfaces were taken by a horizontal plane the curve cut from the sphere would be a circle, and that from the cylinder an ellipse, similar and

equal to that which is the horizontal trace of the latter surface. The two points in which the ellipse and circle meet would be points in the curve of intersection of the surfaces. If, now, an *auxiliary cylindrical surface* were determined, having the circle which is cut from the sphere for its directrix and lines parallel to the generatrices of the oblique cylinder for its generatrices, its horizontal trace would be a circle similar and equal to that which is the directrix of the auxiliary surface, and if generatrices of this cylinder were drawn from the points in which the circular horizontal trace of the auxiliary surface cuts the elliptic trace of the original oblique cylinder, they would cut the directing circle in the same points as the ellipse first mentioned. Hence the following construction:—

(1) Determine a series of sections of the sphere by horizontal planes.

(2) From the centre of one of these circles draw a line parallel to the oblique cylinder.

(3) Find the H.T. of this line, and from that point as centre describe a circle in the H.P. with radius equal to that of the circle cut from the sphere.

(4) From the points in which this circle (which is the H.T. of the auxiliary cylinder) cuts the ellipse, draw generatrices meeting the directing circle, for points in the curve of intersection.

Similarly, by treating the other circles in the same way a number of points in the curve of intersection will be found.

2. *Arrange an oblique cone intersecting a sphere, and find the curve of intersection.*

The construction in this example is similar to that of the preceding problem, except that the auxiliary surfaces are cones, having the same vertex as the given oblique cone, instead of cylinders; the directrices being, as before, the circular sections of the sphere.

NOTE. The constructions given for the two problems of this group apply to the intersection of an oblique cylinder or cone with any surface whose sections by horizontal planes are circles, *e.g.* the *Anchor ring*, or the *Spheroid*, with vertical axis.

GROUP VI.

1. *To find the intersection of two surfaces of revolution when their axes lie in one plane.*

There are two cases:—

α. *When the axes are parallel.*

β. *When the axes meet.*

For α:—Use auxiliary section planes at right angles to the axes.

For β:—Take auxiliary sections of the surfaces by a series of concentric spheres, described from the point in which the axes meet as centre. Every such auxiliary sphere will cut each of the surfaces of revolution in a circle whose plane is perpendicular to the axis of the surface, and the points, in which the pair of circles cut by each sphere meet, will be points in the curve of intersection. The following special exercise is deserving of attention:—

A triangle ABC is the base of a pyramid whose vertex is V. AB = 5 in.; BC = 4 in.; CA = 3 in. The angle AVB = 60°; BVC = 45°; CVA = 40°.

Find the plan of the point V and its height above the base.

(1) Draw the triangle ABC, and on AB describe a segment of a circle containing an angle of 60°, on BC one containing an angle of 45°, and on CA a third containing an angle of 40°. Euclid, Book III. Prop. 33.

If these segments be rotated about the lines AB, BC, CA as axes, surfaces of revolution will be generated whose common intersection will determine V.

(2) Find the intersection of any two of these surfaces thus:—AB and BC being the two axes of the surfaces whose intersection we require, from B with any radius describe a circle cutting the segment described on AB in p, and the segment on BC in q.

If the segment on AB revolve about that line as an axis, the point p will trace a circle, in a plane perpendicular to AB, on the surface of a sphere with centre B and radius Bp. The *plan* of the locus of p is, therefore, a straight line drawn from p at right angles to AB and produced to an equal distance on the other side of that axis.

Similarly, the plan of the locus of q—which also traces a circle on the above-mentioned sphere—will lie in the perpendicular from q to BC.

The point of intersection of the plans of the loci is a point in the plan of the intersection of the two surfaces of revolution.

In like manner any number of points may be found in the plan of the intersection, and the latter line drawn through them.

(3) Find the intersection of the third surface with one of the foregoing. The point v, where the plans of the two intersections cross, is the plan of the vertex V.

It may be remarked that there are two points answering to *V*, one above, and one below the plane of the triangle *ABC*. The latter is neglected.

(4) Complete the plan by drawing lines from *v* to *A*, *B*, and *C*, and find the height of *V* thus :—

Draw a line from *v* perpendicular to one of the axes (say *AB*) and produce it to meet the segment of the circle described on this line in the point *x*. Then *Bx* will be the real length of the edge of which *Bv* is the plan. From these data the height of *V* can be at once determined.

2. *To find the intersection of two surfaces of revolution when their axes are not in the same plane.*

Take a series of sections of both surfaces by auxiliary planes and construct the points in which the curves of section meet for points in the intersection of the surfaces.

NOTE. This problem presents no difficulty except the mechanical one of drawing the curves of section, and to simplify this as much as possible assume one of the planes of projection at right angles to one of the axes, and the other plane of projection parallel to both. The auxiliary section planes may then be taken parallel to the former plane of projection and the intersection deduced without much trouble.

COMBINATION OF CURVED FORMS WITH FORMS BOUNDED BY PLANE SURFACES.

1. *Intersection of a right hexagonal prism standing on its base of* 1 *in. side with a sphere of* 2·5 *in. diameter. One face of the prism making an angle of* 20° *with* xy; *centre of sphere* 2 *in. above H.P. and in plan* ·5 *in. from the plan of the axis of the prism.*

Take auxiliary sections of the sphere by a series of horizontal planes in pairs at equal distances above and below the centre of the sphere.

The plan of the intersection coincides with the base of the prism. The elevation can be found by projection from the points furnished in plan by the auxiliary sections.

NOTE. Any surface of revolution with vertical axis, or oblique cylinder or cone with circular horizontal trace, might be substituted for the sphere in this problem.

2. *Arrange an example of a right square prism, axis horizontal and inclined to V.P., intersecting a right cylinder, axis vertical.*

Divide each edge of the base in plan and in elevation into a convenient number of similar parts, and draw through the points of division a series of lines in each face parallel to the long edges of the prism.

The plan of the intersection coincides with the base of the cylinder. For the elevation, it will be necessary to draw projecting-lines from the points where the parallel lines in plan, and the edges of the prism, cross the circle, to the corresponding lines in elevation. Special attention should be given to the edges of the prism, which should be put in dark or dotted, or partly rubbed out as they require, before joining the points for the curves of intersection.

NOTE. The same construction would apply were the prism oblique.

3. *Arrange an oblique pyramid intersecting a vertical right cylinder.*

Divide each edge of the base of the pyramid as before, and draw the lines to the vertex. The rest of the work is the same as in the preceding problem.

4. *Arrange a right pentagonal prism standing on its base intersecting a horizontal right cylinder.*

Use horizontal lines (generatrices) on the surface of the cylinder as in previous cases of the intersection of the horizontal cylinder.

Pay attention to the *key points* and *critical lines.*

NOTE. The cone might be substituted for the cylinder in this problem, in which case the lines would be drawn from the points of division of the base to the vertex.

5. *Arrange a right pentagonal prism standing on its base intersecting a right cone, axis oblique to both planes of projection.*

6. *Intersection of Cone and Prism.*

Prism, axis horizontal, 4 *in. long, and inclined* 30° *to V.P., bases equilateral triangles of* 2 *in. side, lowest face inclined* 15°, *and lowest edge* ·25 *in. above H.P.*

Cone, axis vertical, 3 *in. long, and passing through the middle point of the axis of the prism, base* 1·5 *in. radius.*

Draw a series of horizontal lines in each face of the prism, as in the prism Prob. 2 above.

Horizontal planes, passing through these lines and cutting the cone in circular sections parallel to its base, may now be taken, and the points in which the plans of the horizontal lines and circular sections of the same levels meet, joined up for the plan of the intersection. The elevation can be completed by projecting.

7. *Arrange a square prism, axis horizontal, intersecting a sphere.*

Same method as in Prob. 6. Any surface of revolution with vertical axis, or an oblique cone with a circular trace, can be substituted for the sphere.

8. *Arrange a square prism, axis horizontal or oblique, intersecting a right cone, axis inclined to both planes of projection.*

Prepare the prism by drawing lines in its faces parallel to the edges, as in other problems, and use section planes containing these lines and the vertex of the cone.

NOTE. An oblique cylinder might be substituted for the cone, but the vertex being at an infinite distance the auxiliary planes would be parallel.

9. *Arrange a sphere intersecting an irregular pyramid.*

Use a series of horizontal auxiliary sections of both forms.

SHADOWS.

In a homogeneous medium light travels in straight lines, called *rays*, which are parallel, divergent, or convergent, according to the source whence the light proceeds and the configuration of the media through which it has passed.

The sun's rays are sensibly parallel; rays from the electric light are divergent, and, for all practical purposes, may be considered as diverging from a point; while rays of light that have passed through a double convex lens, for example, converge towards a point or *focus*.

If an opaque body be placed between a source of light and the object or objects illuminated, some of the rays will be received upon the interposed body and will, consequently, be cut off from illuminating certain parts of the objects beyond. Popularly the opaque body would be said to throw or cast a *shadow*. There is, however, an ambiguity in the term shadow as commonly employed, which it is

convenient, when treating of the projection of shadows, to avoid by suitable nomenclature.

It will be evident after a little consideration that, when the light from a given source is partly intercepted by an opaque body, there is carved as it were out of the surrounding bundle of luminous rays, a central part or core of darkness of a definite form, the surface of which derives its configuration from the interposed body. To the surface which separates this core from the surrounding rays of light the term *shadow-surface* will be given. The line of intersection of the shadow-surface with the forms or surfaces it meets will be called the *shadow-trace.* The figure bounded by the shadow-trace is plainly the *shadow* cast by the interposed body. The *shade-line*, or *line of contact* of the shadow-surface with the form whose shadow is projected, marks the separation of *light* and *shade* thereon. The shade-line is oftentimes an edge or boundary of the form itself.

From these considerations it is apparent that the problem for the student in any given case is to find the shadow-surface, and its intersection with the surfaces on which the shadow falls. Sometimes the shadow-surface is treated as the boundary of a definite form, at others a convenient number of lines (rays) are drawn in the shadow-surface from points in the *shade-line*, and their intersections determined with the forms or surfaces on which the shadow is cast; but in both cases the problem is one of "Sections by Planes" (oblique or otherwise) or of "Intersections."

It may be remarked that the shadows we deal with are purely *geometrical*, and their boundaries, or *traces*, sharp distinct lines. Those actually seen in nature, called *physical shadows*, are more or less hazy about the region of the

shadow-trace, due partly to inflexion and partly to the formation of a *penumbra*—a fringing semi-shadow which makes its appearance when the source of light has an appreciable angular magnitude.

I. Shadows cast by Parallel Rays.

The rays of light are in all cases represented by straight lines given in direction.

One of the parallel rays being determined by its plan and elevation, the shadow-surface is supposed everywhere parallel to the given ray and enveloping the form whose shadow is to be projected.

For example, if the form is a sphere the *shadow-surface* is the enveloping cylinder whose axis is the line drawn from the centre of the sphere parallel to the given ray. The *shadow-trace* is the intersection of this cylinder with the forms or surfaces on which the *shadow* falls. The "circle of contact" is the *shade-line* which separates the part of the sphere *in light* from the part *in shade.*

Taking in lieu of the sphere a right cone in space, axis oblique to the co-ordinate planes, its *shadow-surface* would be partly determined by a pair of tangent planes parallel to the ray, thus:—Draw a line from the vertex of the cone parallel to the given ray and determine two tangent-planes to the conic surface containing this line. The planes so drawn form a portion of the shadow-surface required, and if the plane of the base of the cone were parallel to the given ray, that plane produced would intersect the two tangent-planes in parallel lines, giving rise to a triangular prism as the complete shadow-surface. The intersections of this prism with the surfaces or forms on which the shadow falls will

give the shadow, and shadow-trace, as in other cases. The "lines of contact" of the tangent-planes, and the arc of the base that lies between them on the portion of the conic surface turned towards the light, form the *shade-line*. Were the plane of the base not parallel to the given ray, this part of the shade-line would be treated as the directrix of a cylindrical surface; that is, points would be assumed in it, and lines drawn from them parallel to the given ray, and the intersection of these lines with the objects they meet determined for points in the shadow-trace. Generally, lines drawn parallel to the given ray from assumed points in the shade-line will intersect the plane or curved surfaces on which the shadow is cast in points which determine the shadow-trace. In many cases a few points suffice. For example, in finding the shadow cast on the horizontal plane by a cube, it will be seen that the shadow-surface is bounded by planes parallel to the given ray containing certain edges (those forming the *shade-line*) of the solid, and that the shadow-trace is made up of parts of the horizontal traces of these planes. If, therefore, two lines be drawn from the extremities of one of these edges parallel to the given ray, and their horizontal traces found and joined by a straight line, this line will be one side of the shadow-trace.

Exercises.

1. *The shadow of a cone on a cylinder and of both on the H.P. Cylinder resting on H.P., axis horizontal and inclined at* 30^0 *to V.P., base* 1·5 *in. diameter. Cone resting on H.P., base* 2 *in. diameter, axis* 4·5 *in. long, vertical and* 1·5 *in. from nearest generatrix of cylinder. Rays inclined* 45^0 *and in plan at right angles to the plan of the axis of the cylinder. Show the shade-lines on both surfaces.*

2. *A circular slab* 3 *in. diameter,* ·4 *in. thick, rests symmetrically over the upper base of a right hexagonal prism, axis* 2·5 *in., side of base* 1·5. *Vertical face of prism inclined at* 20° *to V.P. Shadow of the slab on the prism and of both on the H.P. by rays of light making angles of* 45° *with* xy *in plan and elevation.*

3. *A right cone axis* 4 *in. long, base* 2 *in. diameter, resting upon H.P. with the rim of its base touching the V.P. and the plan of its axis making an angle of* 60° *with* xy, *is truncated by a plane at right angles to the axis and* 1 *in. from the vertex. Shadow of frustum by rays of light inclined* 45° *to* xy *in elevation and* 50° *in plan.*

4. *Assume an oblique prism and a sphere, and show the shadow cast by the sphere on the prism.*

5. *A niche is formed, in a vertical wall parallel to the V.P., by a semi-cylindrical recess of* 2·5 *in. diameter capped by a semi-hemispherical dome. Show the shadow in the niche as cast when the rays of light are inclined in plan and elevation at equal angles of* 45° *to* xy.

II. Shadows cast by Rays that meet in a Point.

This division includes the projection of shadows cast by rays *divergent* and *convergent.* The former is the only case that will receive attention, inasmuch as whatever is said concerning the one, is, with very obvious modifications, equally applicable to the other.

The source of light is a point, given by its projections, from which the rays diverge in every direction.

A line passing through the given point and moving

round the body whose shadow is to be projected, generates the shadow-surface and determines the shade-line.

Thus, the shadow-surface of a sphere is an enveloping cone having its vertex in the given luminous point. The "circle of contact" will be the line of shade, and the shadow-trace will be determined in any given case by the methods specially applicable to the intersection of the cone and the particular form or surface on which the shadow is cast.

If the cone selected as an illustration under parallel rays be taken, the work is substantially the same as there given. The two tangent-planes to the cone that pass through the given luminous point constitute a portion of the shadow-surface, the rest of which will be a cone having the luminous point as vertex and a segment of the base as directrix. *Any* required points in the shadow-trace can always be found by lines from the vertex, through points in the shade-line, produced to meet the objects on which the shadow falls.

The shadow-surface of a solid bounded wholly by plane figures will be a more or less irregular pyramid having its vertex in the luminous point. The plane faces of this pyramid will contain certain of the edges of the form whose shadow has to be projected, and in this case, as in the case of parallel rays, the work may sometimes necessitate that two lines only be drawn from the point—one through each extremity of an edge—to determine one side of the shadow-trace.

Exercises.

1. *Arrange a right cone standing on its base and a cylindrical slab resting on H.P. with its axis vertical, and*

show the shadow cast by the cone upon the cylinder, and of both upon H.P., by rays diverging from an assumed point in space.

2. *Assume an oblique plane, a vertical circular disc, and a point in space, and show the shadow of the disc upon the oblique plane as cast by rays of light diverging from the point.*

3. *A hemispherical bowl rests upon the H.P. with the cavity upwards, show the shadow cast inside, the shadow of the hemisphere on the H.P. and the line of shade, when the rays radiate from an assumed point in space.*

4. *Arrange an example showing the shadow of a sphere cast upon an oblique pentagonal pyramid by rays diverging from a point.*

III. Shadows cast by Luminous Bodies of appreciable Angular Magnitude.

For example, suppose the luminous body to be the sun, and the body whose shadow is cast the moon. There are in this case two shadow-surfaces, viz. the two enveloping cones; the trace of the inner surface bounding the true *shadow*, and that of the outer surface bounding the *penumbra*.

The arrangement of a few examples to illustrate the projection of the penumbra must be left to the ingenuity of the student. One that may be suggested, is the shadow of a disc by a white-hot platinum wire of a given length. The source of light in this case being treated as a line of light, show the section of the shadow-surfaces by an oblique plane.

X.

FIGURED PROJECTIONS AND SCALES OF SLOPE.

DEF. The *index* of a point is a number affixed to its projection on a plane to denote the distance of the point from that plane.

A *point in space* is completely determined when its plan and the affixed index denoting the length of its projector are given. Similarly, a *straight line* is determined by its plan and the indices denoting the length of the projectors of any two fixed points in it. The height of any other point in such a line, is, of course, readily determinable by a rabatment of the projecting plane of the line into the plane of projection.

A *minus sign* is prefixed to indices that denote points *below* the plane. Points above the plane are indexed by numbers in the usual way, the positive sign being understood. The index of a point in the plane is of course always zero.

DEF. Points, lines, figures, and forms, determined by one projection and its affixed indices, are said to be given by their *figured plans*.

A *plane* is determined when three contained points, or

two straight lines (meeting or parallel) in it, are given by their *figured plans.* But for convenience in representing indefinite planes by the method of figured plans, recourse is had to a device called the *Scale of Slope,* the nature of which will be readily understood from the following considerations:—

If a pair of horizontal lines be drawn in a given plane, they will both be parallel to its horizontal trace, and, therefore, to one another. Any two such horizontals (one of which might be the H.T. itself), if properly indexed, would determine the plane. A line drawn at right angles to the figured plans of the pair of horizontals might be the plan of a line measuring the inclination of the plane, and from the known indices of the two points where the line of inclination meets the pair of parallels, a number of other indexed points (denoting a series of horizontals) might be at once determined in the line of inclination, and a scale showing heights in the plane constructed. This is what is done in the *Scale of Slope,* which is always drawn as a double line with an interspace of about $\frac{1}{10}$th of an inch. The line on the left-hand side of the scale in the ascending direction should be made thicker than the other, as a convention to assist the eye in reaching the drawings in which Scales of Slope are employed.

A *circle* is completely determined when its plan and the index of its centre are given. Also, a *sphere* is similarly determined when its plan and the index of its centre are given.

A few problems to be worked by the method of figured plans follow. The student who wishes for others will have no difficulty in selecting good examples from among the exercises already given for solution in the usual way.

PROBLEMS.

1. *An equilateral triangle* abc *of* 3 *in. side, is the plan of another triangle whose corners* A,B,C, *are* 1 *in.*, 1·5 *in., and* 2·5 *in. respectively above* H.P.; *determine the Scale of Slope of the plane of the triangle* ABC, *and the index of a point,* P, *in this plane of which the plan,* p, *is the centre of the equilateral triangle.*

Find in *ac* the plan of a point having the same index as *B* (*i.e.* 1·5). A line from *b* through this point will be the plan of a line of level in the plane of the triangle. Draw the pair of lines for the Scale of Slope at right angles to this line. A second horizontal from *A* or *C* will determine another point of known height, whence the Scale of Slope may be completed by divisions showing, for example, differences of level of (say) $\frac{1}{10}$th of an inch.

A horizontal from *p* to the Scale of Slope will ascertain the index of *P*.

2. *To find the index of* P *without recourse to the Scale of Slope.*

Join one of the points, as *a*, with *p*, and produce the line *ap* to meet *bc* in *o*. Find the index of *O* in *BC*, and then of *P* in *AO*, by rabatting the projecting planes of these lines.

3. *To find the true shape of the triangle* ABC.

Rabat the plane about one of the horizontals till parallel to H.P.

4. *Assume two inclined planes by their Scales of Slope, and determine the figured plan of the intersection, and its inclination.*

Draw in the one plane a pair of horizontals through any

two points of division in its Scale of Slope. A similar pair in the second plane, having the same indices as those first drawn, will meet them in points in the plan of the intersection, whence the line may be indexed, and its inclination determined by rabatting its projecting plane.

5. *Given a plane by its Scale of Slope and a straight line by its figured plan, find the point of intersection of the plane and line.*

Assume any convenient plane containing the line, and determine its intersection with the given plane. The point in which the plan of the line meets the plan of the intersection of the planes will be the point required.

6. *Given a regular hexagonal pyramid by its figured plan, and an inclined section plane at* 60^0 *by its Scale of Slope, determine the figured plan and true shape of the section.*

7. *Assume two lines, neither parallel nor meeting, by their figured plans, and determine the surface generated by a third straight line, which moves so as always to meet the two given lines and be parallel to the plane of projection.*

This surface, which is the *Hyperbolic-Paraboloid*, is determined by drawing a series of generatrices from points in one directrix to points in the other having similar indices.

8. *Work a section of the above surface by an inclined plane assumed by its Scale of Slope.*

9. *Work Prob.* 7 *above when the directing plane is not the plane of projection, but is given by its Scale of Slope.*

10. *An ellipse, which is the figured plan of a circle, and an inclined line, are the directrices of a Conoidal surface, the*

plane of projection being the directing-plane, or plane to which the generatrices are parallel; determine the surface, and the section by an inclined plane given by its Scale of Slope.

NOTE. When working Probs. 7, 8, 9, and 10, the student should refer to remarks on "Undevelopable Ruled Surfaces," Chap. VII.

11. *To determine by the method of parallel sections the intersection of two or more forms with plane faces, when the forms are given by their figured plans.*

To work also a section by a given vertical plane.

For the first example take two irregular polygons with several edges of the one cutting those of the other, and assume them to be the bases of two pyramids resting upon H.P. Assume two points for the plans of the vertices and attach indices to denote their heights above the plane of projection. Join the plans of the vertices with the corners of the corresponding bases for the complete plans of the intersecting forms.

To work the intersection:—A little attention bestowed upon the plan and the heights of the vertices will make it evident what faces intersect, and the general direction of the intersections. Sections of any pair of intersecting faces by two horizontal planes will determine two points in the intersection of the indefinite planes of those faces, through which points the line of intersection can be drawn. The horizontal plane of projection itself will serve as one of the auxiliary section planes. Thus, the points in which the polygonal bases meet will be points in the intersection of the faces passing through them. To find another set of points, take an auxiliary section of both forms by a horizontal plane at a convenient height; that is, draw horizontal lines at the same

level in each pair of intersecting faces. The points in which the plans of these lines at the same level meet, will be the plans of the points required.

When two points in each line of intersection are found, the segment of the indefinite intersection that is common to, and limited by the intersecting *faces*, may be put in, and so on till the whole intersection is complete.

NOTE. It is sometimes necessary to produce the sides of the bases, or the lines of level, beyond the limits of the intersecting faces, in order to determine conveniently the required segment of the indefinite intersection.

Work is sometimes saved by taking the horizontal lines in the planes of the faces of the pyramid with the higher vertex, at the same level as the vertex of the lower one.

For another exercise take two pyramids, one with its base resting on H.P. and the other inclined.

The intersection is found, as before, by means of lines of level in the intersecting faces, but in this case the inclined base has to be taken into consideration. Generally it is best to begin with the section of the one pyramid by the inclined plane of the base of the other, and it sometimes simplifies matters if the horizontal trace of the latter form is found before working the rest of the intersection.

To work the section by the given vertical plane it is only necessary to find the indices of the points where the line which is the plan of the section plane cuts the edges and lines of intersection of the forms. The section-plane, with these points of section, can then be rabatted, and the true shape of the section determined.

12. *Find the Scale of Slope of a plane inclined at* 40°,

and the plan and true shape of the section cut by it from the intersecting pyramids in the preceding problem.

13. *Determine the intersection of a right cone and an irregular pyramid, both resting upon* H.P., *by the method of parallel sections, using only the figured plans of the forms.*

14. *A cube, diagonal of the form vertical, stands with its lowest corner on the horizontal plane. Determine the Scales of Slope of the planes of the three faces meeting in that corner.*

XI.

TRIMETRIC METHODS OF PROJECTION.

In the preceding chapter a method is given of working with *figured plans*, by which the necessity for a second projection on a vertical plane is, in a measure, obviated. It is intended to discuss here some developments of this method, one of which is of considerable practical utility from the ease with which it can be applied to the construction of very intelligible drawings.

All forms can be completely determined by referring them to a system of three rectangular planes. Such a system of reference planes intersect in, and are themselves determined by three rectangular axes meeting in a point. Inasmuch as the forms most common in practical constructions are chiefly developed in the three directions of length, breadth, and thickness, it is obvious they might be readily referred to such a system of rectilineal axes, and the lines for their drawings laid off about the projections of these axes on a plane oblique to them. The following considerations will make this clear:—

If the three rectangular axes were projected on a plane oblique to all and differently inclined to each, three scales (one for each axis) might be constructed to show the projected lengths of the unit when set off in the three several directions of the axes. Drawings made with reference to

these axes and their scales, would be true *orthographic* projections, showing at a glance, in one projection, the three principal directions in which the forms projected are developed, and affording a means of readily ascertaining their dimensions. Such drawings might be conveniently called "TRIMETRIC PROJECTIONS." If two of the axes were equally inclined to the plane of projection at angles different from the third, *two* scales only would be required, and the somewhat simplified projection might, in this sense, be called *Dimetric*. In the particular case in which the three axes are *equally* inclined, the projection is *monometric*, or, as it is usually termed, "ISOMETRIC"—only *one* scale being required for the three directions.

For some purposes (*e.g.* the projection of the forms of crystals) it is convenient to regard the intersections of the three planes as *axes of symmetry*. In the forms of the cubic system, for example, since the faces can be disposed symmetrically about three such axes of equal length, any conventional sign denoting one face equally symbolises the whole form, and if the data for fixing one face be given the projection of the complete form can be readily determined. This method is a particular case of *Trimetric Projection*, the application of which to the projection of symmetrical forms will be touched upon briefly under the heading of "AXIAL PROJECTION."

TRIMETRIC PROJECTION.

Before the exercises can be attempted the student must solve the following problem :—

Draw three straight lines, ox, oy, oz, *forming three unequal obtuse angles* xoy, yoz, zox, *at a point*, o, *and assuming these*

lines to be the plans of the three rectangular axes, OX, OY, OZ, *determine the set of trimetric scales corresponding to them.*

Read Prob. XXVII., Chap. III.

It will be necessary first to find the inclination of each axis to the plane of projection. This done, the scales can be readily constructed.

To find the inclination of OX:—Since *OX* is perpendicular to the plane of *YOZ*, the horizontal trace of this plane will be at right angles to *ox*, the projection of *OX*. Draw, therefore, an assumed horizontal trace of the plane *YOZ* at right angles to *ox* and meeting *oy*, *oz*, in *s* and *t* respectively. Similarly, from *s* and *t* draw the horizontal traces *sr* and *tr* of the planes *XOY*, *XOZ*, meeting *ox* in *r*.

Assume the vertical plane containing *ox* to be rabatted thus:—Produce *xo* to meet *st* in *m*, and on *rm* as diameter describe a semicircle. A line at right angles to *rm* from *o* will cut the semicircle in *O'*—the point *O* rabatted—and the angle *O'ro* will be the inclination of *OX* required.

Similarly the inclinations of *OY* and *OZ* may be found.

To construct the trimetric scale of OX:—Draw two lines making an angle equal to the inclination of *OX*: on one of them lay off the divisions of the scale of units employed (inches in the following exercises), and from the points of division draw a series of perpendiculars to the other line. These will determine the divisions of the trimetric scale of *OX*. Work similarly for the others, and finish the scales neatly to show inches and tenths. The scales must be carefully named to guard against the error which would result from interchanging them.

EXERCISES.

1. *Trimetric projection of a brick,* 9 *in. long,* 4·5 *wide, and* 3 *in. thick, to half scale.*

Measure the dimensions (taken from the trimetric scales) along the corresponding axes, and complete by drawing lines parallel.

2. *Trimetric projection of a right cylinder,* 3·5 *in long and* 2 *in. diameter.*

Assuming the cylinder to be circumscribed by a square prism, determine the projection of the latter. Two ellipses inscribed in the trimetric projection of the square ends will be the projections of the circular bases of the cylinder. Two parallel tangents to the ellipses will complete the required projection.

3. *Show the trimetric projection of a circular hole of* 2 *in. diameter bored through the middle of the brick, Exercise* 1., *at right angles to its large faces.*

4. *Trimetric projection of your* T-*square to a convenient scale.*

5. *Trimetric projection of a mortise and tenon joint.*

6. *Trimetric projection of your instrument case with the lid open at right angles.*

7. *A large central semicircular arch-way,* 30 *ft. span and* 40 *ft. high to the springing of the arch, is flanked by two smaller archways—one each side—of* 10 *ft. span and* 25 *ft. to the springing of the arches. The central piers are* 5 *ft. and the outer ones* 2 *ft.* 6 *in. thick. Width at right-angles to*

the plane of its face 25 *ft. Other dimensions and embellishments at pleasure. Show its trimetric projection.*

Scale 10 *ft. to* 1 *in.*

8. *Trimetric projection of a brick with two semicircular grooves of* 1·5 *in. radius cut. in a large face—one running parallel to its long edges through the middle of the face, and the other bisecting the former at right angles. Show the intersection of the grooves. To half scale.*

9. *Trimetric projection of hexagonal right pyramid; axis* 3·5 *in. long, side of base* 1 *in.*

Let *abcdef* be the hexagonal base. Produce two of its parallel sides as *ab*, *de*, and through *c* and *f* draw a pair of parallel lines at right angles to, and meeting the produced sides *ab*, *de*. This forms a rectangle circumscribing the hexagon.

Set off the reduced lengths of the sides of this rectangle along two of the projected axes, as *oy*, *oz*, and complete the parallelogram for the projection of the circumscribing rectangle. The corners of the hexagonal base may now be found, in the sides of the parallelogram, and the projection of the base drawn in.

For the vertex, draw from the centre of the parallelogram a straight line (the projection of the axis) parallel to *ox*, and make it equal to 3·5 in. on the scale of *OX*. Join the vertex with the corners of the hexagonal base for the complete projection of the pyramid.

NOTE. By the the device of an enveloping rectangle, or, more generally, by the aid of rectangular ordinates, any plane figure whatever, or any prismatic or pyramidal form,

may be projected as above. The general cases will be, however, treated under *Trihedral Projection* below.

10. *A figure formed by describing two semicircles on two parallel sides of a regular hexagon of* 1·25 *in. side as diameter —the concavity of the semicircles being turned in the same direction—is the base of a prismatic form* 4·5 *in long. Show its trimetric projection.*

11. *Show the Trimetric projection of your instrument case when the lid is open at an angle of* 60°.

ISOMETRIC PROJECTION.

Isometric Projection has been already defined as the particular case of Trimetric Projection that results from the three axes being equally inclined to the plane of projection. In this case it is clear that the obtuse angles formed by the meeting of the plans of the three axes at the point, *o*, are all equal—*i. e.* are each = 120°.

We may therefore begin by drawing the plans of the three axes at equal angles of 120°, and proceed to determine the scale of one of the axes as for the trimetric scale above. This scale, which also serves for the other two axes, is called the *Isometric Scale.*

The ratio of a line, measured along an isometric axis, to its isometric projection may be readily proved to be as $\sqrt{3}$ is to $\sqrt{2}$. We may, therefore, avail ourselves of this established ratio and construct as follows the

Isometric Scale:—Draw a right-angled triangle having one side 1 in. long and the other equal in length to the diagonal of a square of 1 in. side. The diagonal of a

square whose side is unity is equal to $\sqrt{2}$, and the hypotenuse of a right-angled triangle with one side equal to unity and the other to $\sqrt{2}$, will be equal to $\sqrt{3}$ (Euclid 47, Bk. I.). That is, the hypotenuse of this triangle is to its longest side as $\sqrt{3}$ is to $\sqrt{2}$. Therefore, real lengths may be measured along line $\sqrt{3}$ and the corresponding isometric lengths found in line $\sqrt{2}$ by drawing perpendiculars to that line from the points set off in the hypotenuse $\sqrt{3}$, or a scale of isometric inches may be similarly constructed.

Isometric being but the most simple form of Trimetric projection, it is unnecessary to give further exercises. All those given under Trimetric projection, or any that the student has omitted, may, however, be worked here at his own or the teacher's discretion.

AXIAL PROJECTION.

The term Axial Projection is used to denote a trimetric method which lends itself with great facility to the projection of symmetric forms bounded by plane faces, such as the forms of crystals.

The investigation of the geometrical properties of these forms, and of the number and position of the axes of symmetry to which their faces can be most conveniently referred, is the special province of *crystallography*. The principles on which they are projected fall, however, within the scope of Descriptive Geometry, while many of the forms have an interest attaching to them quite apart from their crystalogical significance.

OX, *OY*, *OZ*, being the three rectangular axes as

before, it is evident that an indefinite plane will in general meet them in three points, which may be called the *axial traces* of the plane. Let α, β, γ denote the distances of the points of section or axial traces along OX, OY, OZ respectively. A plane, relatively to the three axes, is obviously completely determined by its axial traces. The plane of the face of a form referred in this way to three rectangular axes can therefore be expressed by the ratio of the parameters of the face:—$\alpha : \beta : \gamma$.

The student will observe that it is in general convenient to treat the axes as three lines drawn *from* the point, O, to indicate three directions in which certain measurements are to be set off, but for the present purpose the axes must be produced *through* the origin, O, on both sides. The system of intersecting planes that give rise to the indefinite axes XX', YY', ZZ', in question, form eight similar trihedral angles, whence it follows that for every plane determined by its axial traces in one of these trihedral angles there will be seven other planes similarly related to the axes—one in each of the other seven trihedral angles.

If the three fundamental lines, ox, oy, oz, used as the projection of the axes in previous problems, be produced and lettered xox', yoy', zoz', any required trihedral angle can be specified by the extremities of the semi-axes, as xyz, yzx', &c.

Exercises.

1. *One face of a form symmetrical about three equal rectangular axes is given by the formula* $\alpha : \beta : \gamma = 1 : 1 : 1$; *determine its axial projection.*

The projections of the three indefinite axes and the set of scales may be determined once for all, as in the case of the trimetric exercises above.

The greatest number of faces similar to the one given that can be disposed about the axes is eight—one in each trihedral angle. The faces are equilateral triangles, and the form is, therefore, a regular octahedron.

To draw the projection it is only necessary to cut off, by means of the scales, equal lengths on each axis from O, on both sides of that point, and join up the points so found for the edges of the octahedron. For example, let 1, 2, 3, be the plans of the points on the semi-axes ox, oy, oz, respectively, then if 1 be joined with 2, 2 with 3, and 3 with 1, the triangle 123 will be the projection of the face situated in the trihedral angle xyz. Similarly the other faces may be drawn.

2. *Determine the form produced by the intersection of the indefinite planes of the alternate faces of the octahedron drawn in the preceding problem.*

Lines through the points 1, 2, 3, parallel respectively to lines 23, 31, 12, will give the intersections of the face 123 with the alternate faces of the trihedral angles $xy'z'$, $x'yz'$, $x'y'z$. These intersections are three edges of one face of the tetrahedron—which is the form produced. Similarly the other edges can be found.

3. *Axial projection of a symmetrical form one face of which is given by the formula* $\alpha : \beta : \gamma := \infty : 1 : \infty$. The sign ∞ = infinity.

The faces of this form are six in number—each meeting one axis at unit distance from O and parallel to the other two. The form produced is, therefore, a cube.

The intersections of the planes being parallel to the axes, no difficulty will be experienced in drawing the form.

4. *Axial projection of the form produced by planes parallel to one axis and meeting the other two at the unit distance:* $a : \beta : \gamma = \infty : 1 : 1$.

This form, which is the Rhombic Dodecahedron, will serve as an additional exercise. There are several others, the most general of which is a forty-eight-faced form having the formula $m : 1 : n$, but their discussion belongs more to crystallography than to Descriptive Geometry.

TRIHEDRAL PROJECTION.

Hitherto all forms have been regarded as trimetrically developed about rectangular axes in three directions from their origin *O*. But the most general aspect in which a problem can present itself is when the position of the form is not referred directly to the axes but to the three intersecting planes. A *Trimetric* projection of this kind would also be *Trihedral*, and the latter term serves very conveniently to denote this most general aspect of the trimetric method.

As was before observed, three indefinite rectangular planes intersect in eight similar trihedral angles, but in order to avoid the necessity for a cumbrous system of signs to specify which of these trihedral angles is meant, it will be best to represent all points, lines, figures, and forms in one of these angles only, viz. that formed by the planes *XOY*, *YOZ*, *ZOX*, and limited by the semi-axes *OX*, *OY*, *OZ*. For the purposes of construction the planes may of course be considered as indefinite.

PROBLEMS.

1. *To determine a point,* P, *in space trihedrally.*

Let *a*, *b*, *c*, be the respective distances of *P* from the planes *YOZ*, *ZOX*, *XOY*. Measure the trimetric length of *a* from *o* along *ox* and from the point thus found draw a parallel to *oy* equal to *b* as shown on the trimetric scale of *OY*. A parallel to *oz* from the point last determined, equal in length to *c* on the scale of *OZ*, will give *p*, the trihedral projection of *P*.

2. *Given* p, *the trihedral projection of* P, *and* c, *the distance of* P *from the plane* XOY, *find* a *and* b, *its distances from planes* YOZ *and* ZOX.

Show also the trihedral projections of the orthographic projection of P *on each of the reference planes.*

THEOREM. A point in space is completely determined when its trihedral projection and its distance from one of the three reference planes are given.

The particular trihedral angle in which *P* is situated will depend on the position of *p* and the length of *c*. The following construction although perfectly general, if the student interpret from his drawing the direction in which the parallels are to be drawn from *p*, is intended to apply only to the one trihedral angle used before :—

Draw from *p* a parallel to *zo* equal to the trimetric length of *c*. The extremity of this parallel is the trihedral projection of the orthographic projection of *P* on the plane *XOY*. Two parallels to *xo* and *yo* from the extremity of the parallel first drawn, meeting *oy* and *ox* respectively, determine *a* and *b*.

From a and b thus found the other required projections of P can be readily determined.

NOTE. The student will observe that a line is determined trihedrally by two contained points and a plane by three, but, more generally, by their trihedral traces.

DEF. The trihedral traces of a line are the points in which it pierces the planes of reference.

DEF. The trihedral traces of a plane are the lines in which it intersects the planes of reference.

It is obvious that the trihedral traces of a plane pass through the axial traces of the same plane.

3. *Given the data that determine the trihedral projections of two points,* P *and* Q, *in space find the trihedral traces of the line joining them.*

Assuming that the three reference planes are limited by the semi-axes OX, OY, OZ, it is plain a straight line can have but two trihedral traces and these will always suffice to determine it. Unless, however, the line be parallel to the third plane the two will intersect if produced, and, for the purposes of construction, this third trace might be found useful.

It will be easily seen from inspection in which plane there is no trace. This we will suppose to be the plane ZOX. Through the projections of P and Q on this plane draw a straight line (that is the projection of the line PQ on plane ZOX), and produce it to meet oz and ox in l and m. Parallels from l and m to oy will meet the trihedral plan of the line PQ produced in the required traces of the line.

NOTE. The trihedral traces of a plane containing three given points will of course pass through the traces of the lines joining them two and two. Hence the traces of such a plane might be readily determined.

The foregoing problems are of a somewhat general character and are offered merely as illustrations of a method of projection that has never yet been sufficiently worked out. To the student who has leisure, and the inclination, to think out some of its applications for himself they will, it is hoped, suggest a method that promises some interesting results.

XII.

MISCELLANEOUS PROBLEMS.

1. *Plan and elevation of a cube of* 3 *inches edge, when three of its corners are* 1, 1·7, *and* 3 *inches, respectively, above the horizontal plane.*

Refer to Chapter III. Problem 21.

Take one inch to equal 10 units.

(1) Draw the square $ABCD$, and index A as 1 inch or 10 units, B as 17 units, and C as 30 units, that is, the respective heights to which these points *are* to be lifted.

(2) To find a 'horizontal,' in order that the square may be folded about it into the required position :—

If A and C, which are to be at heights of 10 and 30 units, be joined, there will be a point in the line AC 17 units high, that is, at the same level as point B. Find this point, M, by dividing the line in due proportion, and the line drawn through B and M will be the horizontal required, at level 17 units or 1·7 in. above the horizontal plane.

(3) A vertical plane at right angles to plane of face $ABCD$ may now be assumed by taking a line at right angles to BM, for the intersection, xy, of this vertical plane

and the horizontal plane at level 17 units. The vertical trace of plane of face *ABCD* may be easily determined because a' and c' will, from the position of the assumed vertical plane, be in the vertical trace, and their heights are known.

(4) The square, *ABCD*, is one face of the cube rabatted about the horizontal *BM* into a horizontal plane at level 17 units. The rest of the work is similar to that in former problems.

NOTE. The heights of the three points determine the inclination of two adjacent edges of the square face; hence, another construction based on Chap. v. (see Prob. 7) may be used.

2. *Plan and elevation of a tetrahedron, edge* 3 *inches, when the three corners of its lowest face are* ·8, 1·5, *and* 2·2 *inches above the horizontal plane.*

3. *A right pentagonal pyramid, side of base* 1·5 *in., axis* 3·5 *in., has its vertex* 3 *in. high and the extremities of one edge of the base* 1 *in. and* 1·25 *in. high respectively. Plan and elevation.*

Prob. 26, Chapter III. required.

4. . *Plan and elevation of a cube of* 3 *inches edge, when the indefinite plans of three adjacent edges meeting in point* A *are given.*

Compare Chapter III. Problem 27, and Problem 1, page 216, "Trimetric Projection."

(1) Draw the three given lines as *ab*, *ac*, and *ad* forming three obtuse angles at *a*. Since the projections of a perpendicular to a plane are at right angles to its traces, a line drawn at right angles to any one of these edges, as to *ab*, will

be a horizontal in the plane containing the other two. Draw such a line to cut *ac* in *p* and *ad* in *s*. Then the angle *pas* is the plan of a right angle the plane of which may be rabatted about the horizontal *ps* by bisecting *ps* and describing a semicircle on it. The angle in a semicircle being a right angle, point *A* when rabatted will be in the intersection of the semicircle with *ab*. If *p* and *A*, and *A* and *s*, be joined, the edge of the cube, 3 inches, can be measured from *A* on these lines *Ap* and *As*, or the same produced, and the square can be completed.

(2) The plan and elevation of the whole may be finished by taking a vertical plane at right angles to the horizontal *ps* and working from the square rabatted.

5. *Given an oblique plane by its traces, equally inclined at angles of* 40° *to* xy. *Determine the plan and elevation of a tetrahedron,* 3·5 *in. edge, resting upon it. No edge of tetrahedron to be horizontal.*

Determine the inclination of the plane, Chapter III. Problem 5. Take a new ground line at right angles to the horizontal trace, set up the vertical trace in this new vertical plane, complete elevation and plan therein, and thence derive the elevation upon the first or given vertical plane.

6. *An oblique pyramid stands on its base, which is a regular hexagon* ABC...F *of* 1·5 *inches side.* V *is the vertex, the face* VAB *is inclined at* 75° *to the base,* VBC *at* 65°, VCD *at* 60°. *Draw the plan of the solid.* Science Examination, 1870.

(1) Draw the hexagon.

(2) Determine the traces of plane 75°, taking the *xy* at right angles to the horizontal trace.

(3) The same with plane 65°.

(4) In each of these planes determine a horizontal line at the same height. The intersection of these lines gives a point, *p*, in the plan of the intersection of the two planes.

Another point in the same intersection is *b*, and the line *bp* being the plan of one edge of the pyramid contains *v*.

(5) Work similarly with planes 65° and 60°, and the edge *cv* is determined, also containing *v*. The point *v* is fixed by the intersection of the lines *bpv* and *cv*.

7. *Given the plan of a sphere, required to determine the plan of a cube inscribed in it*, a, *the plan of one of its corners, and the direction*, ab, *in plan of one edge being also given.*

(1) To find the true length of the edge of the inscribed cube. On the diameter of the circle which is the plan of the sphere, mark off one-third of its length from one extremity, and from the point so found erect a perpendicular to meet the circumference. Join the extremities of the diameter to this point in the circumference. The shorter line gives the length of the edge of the inscribed cube, whilst the longer gives that of the edge of the inscribed tetrahedron. *Or*, by computation, if *d* equals the diameter of the sphere in inches, and *e* the required edge of the cube, then

$$\sqrt{3} : \sqrt{1} :: d : e.$$

(2) Make an elevation of *AB* on *ab* used as a line of level at the height of the centre of the sphere, and determine *b*. To do this, describe a circle on the part of *ab* produced which cuts the circle forming the plan of the sphere. Erect a perpendicular to *ab* from point *a* to meet the circumference. This gives *a'*. From *a'* measure *AB* as a chord.

This gives b', and from b' a perpendicular to ab—which being used as a line of level corresponds for this work to a ground line—determines b.

(3) A line drawn from b through the centre o of the sphere with the distance from b to o measured on it from o, will give the plan of g, another point in the face $A...G$.

(4) Determine by its traces the plane containing aa', and perpendicular to ab, $a'b'$. This will be the plane of the face $A...G$ of the cube. To do this, through point a' draw the vertical trace at right angles to $a'b'$, and through m, where this vertical trace crosses ab, the line of level, draw horizontal trace at right angles to ab.

(5) Rabat this plane and with it points aa', gg'. Join AG which will be a diagonal of this face of the cube rabatted. Complete the square, and therefrom determine its plan by rotating the plane back to its original position.

(6) The edge ab being already drawn, parallels to ab, and of the same length, through the points $a...g$ will complete the plan of the solid.

8. *Plan and elevation of a tetrahedron inscribed in a sphere of* 2 *in. radius. Two adjacent edges of the solid inclined at angles of* 20° *and* 30° *respectively. Centre of sphere* 3 *in. above horizontal plane.*

(1) Find edge of tetrahedron inscribed in a sphere of 2 inches radius. See preceding Problem.

NOTE. The edge of tetrahedron is a diagonal of face of inscribed cube.

(2) Draw an equilateral triangle with sides equal to edge of inscribed tetrahedron, and determine the diameter of a circle circumscribing this triangle. Set off the diameter of this circle as a chord of a great circle of given sphere, and find the length of a perpendicular from the centre of great circle to the chord drawn. Call Ox the length of this perpendicular; then Ox is the length of a perpendicular from centre of given sphere to plane of face of inscribed tetrahedron.

(3) Determine on a convenient part of the paper the plane of a face of any tetrahedron having two adjacent edges of that face inclined as above. Chapter III. Problem 20.

(4) Draw from centre, O, of the given sphere, a line perpendicular to the plane determined in (3) and cut off on this line from the centre of given sphere a length equal to Ox. A plane through the point x at right angles to the perpendicular Ox, will be parallel to the plane of the face of the auxiliary tetrahedron first drawn, and will also be the plane of one face of the solid required.

(5) Rabat this plane with the circle in which it cuts the given sphere.

(6) Inscribe in the rabatted circle the triangular face of the required tetrahedron. To do this:—draw three tangents to the circle parallel to the three sides of the triangular face of the auxiliary tetrahedron first drawn for the purpose of determining the plane of a face of that solid. The three tangents constitute an equilateral triangle circumscribing the circle. Draw lines from the three corners of this figure to the centre of the circle, and join the points

where these lines cut the circumference. This will be another equilateral triangle with sides parallel to the one inscribed in the rabatted circle, and will be, moreover, the rabatted plan of the face of the required inscribed tetrahedron, from which the projections can be at once completed.

9. *The centre,* O, *of a sphere of* 2 *inches radius is* 3 *inches above the plane of the paper. A point,* a, *on the surface,* 4·25 *inches high, is the plan of one corner,* A, *of an octahedron inscribed in the sphere. The edge* AB *is inclined* 45°. *Draw the plan and an elevation on a vertical plane inclined* 30° *to the line* ao.

(1) Determine the edge of the octahedron. This will be equal to the side of a square inscribed in a great circle of the given sphere.

If r = radius of sphere, x = side of inscribed octahedron, then

$$r : x :: 1 : \sqrt{2}.$$

(2) Make plan and elevation of sphere, and determine the plan and elevation of A on its surface. To do this:—draw, in the elevation, a line, at the height of 4·25 inches, parallel to xy and cutting the elevation of the sphere. This may be considered as the vertical trace of a horizontal plane passing through sphere at the assumed level. The section of the sphere by this plane will be a circle, shown in true form in plan, and as a straight line in the trace of the plane in elevation. The point A is on the circumference of the circle constituting the plane section of sphere, and its projections a, a' are on the corresponding projections of the section. To find a, draw a line from o, the plan of the

centre of the sphere, making an angle of 30° with xy, and produce it to meet the circle which is the plan of the section. The point of intersection is a. a' is at once found by projecting a to the straight line drawn at 4·25 inches above xy.

(3) Make A the vertex of a right cone, axis vertical, generatrix inclined 45° and equal in length to the side of the octahedron (1).

The base of this cone in elevation is a straight line and in plan a circle, with radius equal to that of the given sphere, described from the point a as centre. The section of the sphere by the plane of the base of the cone produced, is another circle. Show the latter in plan. Either of the two points where the plan of the circular base of the cone meets the plan of the circle cut from the sphere by the plane of the base of the cone produced, may be taken as the plan b of the other extremity B of the edge AB.

(4) Determine b' and draw ab, $a'b'$. These are the projections of the edge AB.

(5) Determine the plane containing the straight line AB and the centre of the sphere O. Rabat it and complete the square $ABCD$, which will be inscribed in a great circle of the sphere.

(6) Determine the plan $abcd$, and draw a perpendicular to the plane of $ABCD$ from O, meeting the surface of the sphere on both sides of the plane at P and R.

(7) Complete the plan of the solid by joining p with $abcd$ and r with the same points, and the elevation by joining p' and r' with $a'b'c'd'$.

10. *A parallelogram, sides* 3 *inches and* 2 *inches, included angle* 60°, *is the plan of a square. Determine the side of the square and the inclination of its plane.* Science Exam. Hon. 1871.

(1) Draw parallelogram and inscribe in it an ellipse.

This ellipse will be the plan of a circle inscribed in the square of which the parallelogram is the given plan, and which circle will necessarily lie in the same plane with the square; that is, in the plane whose inclination we have to find.

(2) Determine the major and minor axes of this ellipse.

The former is a horizontal line lying in the plane of the square, and is therefore the diameter of the circle of which the ellipse is a plan. This latter property carries with it two others, viz. that of being the diameter of the circle inscribed in the square whose side is required, and also that of being *equal* to that side.

(3) To determine the inclination of the plane, it is only necessary to make a projection of the minor axis on a vertical plane assumed at right angles to the major axis. The plane of the square and the assumed vertical plane are perpendicular, hence, the elevation of the minor axis of the ellipse will fall in the vertical trace of the plane of the square, and its angle of inclination with xy will therefore be the same as that of the plane.

The student will require no further assistance in determining this plane, or in rabatting it with its contained square for the purpose of verifying the side found from the major axis of the ellipse.

11. *A right hexagonal pyramid,* 5 *inches high, has its axis inclined* 50° *and one diameter* AD, 4 *inches long, of its hexagonal base inclined* 25°. *Draw plan and elevation, and determine the circumscribing sphere.*

(1) If the axis is inclined 50°, the plane of the base will be inclined 40°. Determine this plane by its traces, and draw plan and elevation of the solid resting on it. Refer to Chapter IV. Problem 1.

(2) To determine the sphere. Rabat the plane containing the vertex and a diameter of the base of the pyramid. The horizontal traces of the two slant edges of the pyramid springing from this diameter to the vertex will be in the horizontal trace of plane required. Determine the centre of the triangle formed by this diameter and the slant edges of solid thus rabatted, and describe circle therefrom passing through the three corners of the triangle. This gives the rabatted centre of the required sphere and its radius, whence the required projections of the circumscribing sphere can be readily determined.

12. *Plan and elevation of a pentagonal pyramid when one face is vertical and one long edge of that face inclined at* 50°. *Side of base* 1·25 *in., axis* 3·5 *in.* Science Exam. Hon. 1872.

(1) Determine angle θ between base and one face of pyramid.

(2) Determine a line in the vertical plane of projection inclined 50°, and on this line draw a triangular face of the pyramid, one long edge of that face lying in the line.

If the vertical face of the pyramid be assumed parallel to the vertical plane of projection, the triangular face of the pyramid drawn as above will be an elevation of a face of the

form, and the plan of this face will be a straight line parallel to xy at any chosen distance in front of the vertical plane.

(3) Assume two points in the short side (i. e. the base), or short side produced, of the triangular face of the pyramid, and make them the vertices of two right cones having their bases in the vertical plane and generatrices inclined to bases at θ degrees.

(4) Draw the tangent plane to these cones. The vertical trace will be a line tangential to the circular bases, and the horizontal trace a line drawn through the horizontal trace of the short side or base of the triangular face of the pyramid and the point where xy is met by the vertical trace.

(5) Rabat this plane and the short side of the triangular face with it. On this side rabatted describe the pentagon which is the base of the pyramid, and determine plan and elevation therefrom.

The projections can be at once completed by joining the proper points, all of which are thus determined.

13. *A right pyramid, having a hexagon of* 1·25 *in. side for its base, and an axis of* 3·5 *in., lies with one edge on the horizontal plane, and a face containing that edge is inclined at* 25°. *Draw plan and elevation.* Science Exam. Hon. 1870.

(1) Determine the angle α° between the base and one face of the pyramid.

(2) Determine plane of 25° by its traces.

(3) Rabat this plane and draw in it a triangular face of the pyramid with one of its equal sides lying in the horizontal trace, and thence determine plan.

(4) Produce the plan of the base of the triangular face and rabat it.

NOTE. The line thus rabatted will be the produced base of the isosceles triangle first drawn.

(5) Describe a circle of any convenient radius touching this line and mark, P, the point of contact. Assuming this circle to be the base of a cone with generatrix inclined α^0, find the vertex, V, and lift the plane, with the cone on it, back to its former position.

(6) Find the projections, pp', vv', of PV, and the horizontal trace of a straight line drawn from V to P.

The line drawn through this trace and the point where tangent line to the rabatted base of the cone at point P meets the horizontal trace of plane 25^0, will be the horizontal trace of a plane inclined α^0 to this plane, and will contain the base of the triangular face of pyramid determined in (3), and hence is the plane of the base of that pyramid. This plane can now be rabatted and the projections determined as in previous problems.

It may be remarked that the line PV is that in which the plane of the base of the pyramid touches the cone.

14. *The two extremities* A, B, *of one edge of a cube are at heights of* ·4 *in.*, 1·6 *in. above the paper. The centre*, O, *of the solid is at a height of* 1·4 *in. Draw the plan of the solid. Edge* = 3 *inches.* Science Exam. Hon. 1870.

The section of the cube made by a plane passing through the edge AB and centre O of the solid is a rectangle, two sides of which are edges of the cube and two diagonals of the faces.

By the conditions, the heights of three points—two extremities of one short side, and the centre—of this rectangle are given.

(1) Determine plan and elevation of this rectangle as in Problem 1.

(2) Determine a plane at right angles to one of the short edges of the rectangle and containing one of the long edges of the latter. This condition is complied with if the plane be drawn through one extremity of a short side and perpendicular to it. Converse of Chapter III. Prob. 13.

Or, the horizontal trace of the plane may be drawn at once through the horizontal trace of one long side of the rectangle at right angles to the plans of short sides of the latter.

NOTE. This trace will be the point where the plan of the long side produced meets the horizontal trace of the plane of the rectangle.

(3) Rabat the plane thus drawn, which is the plane of one face of the cube, and with it the long side of the rectangle.

The latter being a diagonal of a face of the cube, the plan of the solid can be completed therefrom as in previous problems.

15. *An irregular pyramid has for its base a triangle* ABC*:* AB = 3 *in.;* AC = 3·5 *in.;* BC = 4 *in.: the plan* d *of the fourth corner projected on the plane of the base is* 2 *in. from* A, 1·5 *in. from* B*; the true length of the remaining edge* CD *is* 3·7 *in. Draw the plan of this pyramid, when standing on its base, and an elevation on a plane parallel to the edge* AD. *Determine the lengths of the edges* AD, BD, *and the height of* D *above the base.*

α. *Determine the inclinations of the faces* ABD, ACD, *to the plane of the base.*

β. *Draw a plan of this pyramid when the edge* CD *is vertical.* Science Exam. 1868.

(1) Draw the triangle *ABC*. The point *d* will be found by describing a circle of 2 in. radius from *A* as a centre and another of 1·5 in. radius from *B*; the point where the circles intersect on that side of *AB* nearest to *C*, is the point required.

(2) Join *dC*, and from *d* draw a perpendicular to this line. A circle of 3·7 in. radius described from *C* as a centre will cut this perpendicular at *D*, and *Dd* will be the height of the vertex of the pyramid.

(3) To determine the lengths of the edges *AD*, *BD*, draw from *d* lines perpendicular respectively to *Ad* and *Bd*. Make these perpendiculars equal to height of vertex = *dD*, and complete the right-angled triangles *AdD*, *BdD*; the hypotenuses *AD*, *BD*, of these triangles are the edges required.

(4) For the required elevation, *xy* must be taken parallel to the plan *dA* of the edge *AD*. The rest of the work presents no difficulty.

NOTE. The elevation of *AD* on this vertical plane is in true length.

(5) For the inclination of the faces *ABD*, *ACD*, elevations may be made on vertical planes at right angles to the lines *AB* and *AC* respectively.

(6) For the plan (β) make an elevation on a plane parallel to *CD*, and from this elevation deduce the plan required, by taking *xy* at right angles to the elevation *c'd'* of *CD*.

16. *A tetrahedron, of 3 in. edge, stands with one corner on the horizontal plane, so that the plans of the two edges meeting at that corner are 1·9 in. and 1·5 in. respectively. Draw the plan of the solid.* Science Exam. Hon. 1871.

(1) Determine the plane of the face containing the edges whose lengths in plan are given.

To do this:—draw a line and make it equal in length to either of the plans of the given edges. Show an elevation of this edge on a parallel vertical plane. For this purpose the *xy* must be taken parallel to the line drawn, and should be assumed at a convenient distance therefrom. An elevation on this plane will show the edge in true length, i.e. 3 inches.

This determines one edge of the solid. The other is of the same real length and inclined at a constant angle of 60° to the first. It is plain, therefore, that the second will lie somewhere on a right conical surface having the first for an axis and the second for a generatrix; that is, on a cone whose vertex is at the point where the first edge meets the horizontal plane,—vertical angle 120°, generatrix 3 inches, and axis coincident with the edge whose projections we have found.

Determine plan and elevation of this cone. The elevation of the base will be a straight line bisecting at right angles the vertical projection of the edge first found, and the plan, an ellipse, having its centre at the middle point of the plan of that edge.

Describe a circle from the vertex of the cone in plan with a radius equal to the length of the plan of the second edge; and from either point common to circle and ellipse draw lines to the two extremities of the first edge.

The triangle thus drawn is the plan of the face of the tetrahedron containing the two given edges. From this the plane of the face can be at once drawn.

(2) Rabat this plane and complete plan of tetrahedron as in previous examples.

17. *An irregular triangular pyramid has its edges* AB = 2; AC = 2·5; BC = 3; AD = 3·33; BD = 3·45; CD = 3·84 *inches. Determine the plan of the sphere circumscribing this solid when the plane of the face* ABC *is inclined* 40° *and one edge* AB *of that face* 30° *to the horizontal plane.*

(1) Determine the plane of 40° and the line inclined 30° lying in it.

(2) Rabat the plane with the line of 30°, and on the line so rabatted describe the triangle *ABC.*

(3) On the side *AB* describe another triangle *ABD.* Imagine the triangle *ABD*, which is a face of the pyramid, to revolve about *AB* as an axis; the point *D* will describe a circle in a plane perpendicular to *AB* and the locus of *D* about the axis *AB* will be, in plan, a straight line drawn from the vertex *D* of the triangle *ABD* at right angles to *AB.*

Draw the plan of this locus and produce it indefinitely.

(4) On the side *BC* describe a triangle *BCD.*

A line from *D* perpendicular to *CB* will be the plan of the locus of *D* about *CB* as axis.

Draw this line and produce it to cut the line determined in (3).

(5) The point *d* where the loci intersect, in plan, is the plan of the vertex *D.* The height of this point can be readily determined by an elevation on a vertical plane parallel to one of the slant edges.

The plan and elevation of the solid on the plane of 40° present no difficulties. See Chapter IV.

The four points *A*, *B*, *C*, *D*, situated at the solid angles of the pyramid, are those through which the spherical surface has to pass, and the centre of this sphere is found as in Problem 1 on "The Sphere," Chap. VII.

18. *Draw a triangle* ABC *having* AB = 2·7, BC = 3·2, AC = 2·3 *inches. The points* A, B, C, *are the stations of three observers who at the same moment take the altitude of a point* P *in space. The angle by* A *is* 33°, *by* B 37°, *and by* C 46°. *Show the plan of the point and find its height in inches above the plane of* ABC. Science Exam. Hon. 1872.

The point *P* is the common intersection of three inverted cones with axes vertical, vertices at the points *A*, *B*, and *C*, and generatrices inclined to the planes of the bases at angles of 33°, 37° and 46° respectively.

To work the problem:—

(1) Make elevations of the cones on a vertical plane, and across the elevations draw a series of lines, at convenient distances apart, each parallel to *xy*. These lines are to be regarded as the vertical traces of a series of horizontal planes at the levels shown by the lines in elevation.

(2) The sections of the cones by the horizontal planes are circles, which are shown in true form in plan. Determine the plans of the sections of any two of the cones. The points where the circles which are the plans of the sections of the two cones under consideration by the same horizontal planes cut one another, are points in the plan of the intersection of those cones, and a curve drawn through these points will be the plan of that intersection.

Similarly, the plan of the intersection of one of the above cones with the third can be found.

The point where the two intersections in plan cross is the plan, p, of the point P.

(3) The elevation p', and therefore the height of P, can be easily determined thus: From the plan of the vertex of one of the cones, as A, describe a circle to pass through p. This circle is the plan of a section of the cone whose vertex is A by a horizontal plane passing through P. A line at right angles to xy and a tangent to the circle will cut the slant side of the cone in elevation at a point the height of P above xy.

19. ABCV *is a pyramid of which* ABC *is the base.* AB = 3·5 *in.*, BC = 3 *in.*, $CA = 2{\cdot}75$ *in. The vertex, V, is* 4 *in.*, 4·25 *in., and* 4·75 *in. from A, B, and C, respectively. Determine the inscribed sphere.*

NOTE. Find the vertex as the common intersection of three spheres of 4 in., 4·25 in., and 4·75 in. radii, described from the centres A, B, and C, respectively. (Compare Prob. 17 of this Chap.)

The centre of the inscribed sphere will be the common intersection of three planes—two bisecting any two of the three dihedral angles formed by the slant faces of the pyramid, and the third bisecting one of the dihedral angles formed by the plane of the base and a slant face. The radius will be determined by a perpendicular from the centre to one of the faces.

20. *Assume three intersecting planes and determine a sphere of* 1 *in. diameter touching them.*

The centre of the required sphere will be the common intersection of three planes, each ·5 in. distant from, and parallel to, one of the three touching planes.

21. *Plan and elevation of a right pentagonal pyramid under the following conditions:—Plane of face* VAB *inclined* 50° *to horizontal and* 75° *to vertical planes, one long edge* VA *of face inclined* 25° *and lowest corner,* A, 1 *inch above the horizontal plane. Other dimensions at pleasure.*

(1) Determine angle α between the base and one of the faces.

(2) Determine the face *VAB* lying in a plane inclined 50° to horizontal and 75° to vertical planes. Chapter III., Probs. 7 and 19.

(3) Determine a plane containing the edge *AB* of the face *VAB* and inclined to the plane of that face at the angle α. Chapter III., Prob. 15, *converse.*

(4) Rabat the plane determined in (3), and on the edge *AB* contained therein describe the pentagonal base of the given pyramid. Determine the projections of the base and join *abcde*, plans of the corners of the base, with *v*, and *a'b'c'd'e'* with *v'*, for the complete projections of the solid.

22. *A right cone, base* 1·5 *in. diameter, axis* 3 *in. long, rolls over the convex surface of another right cone standing upon a circular base of* 4·5 *in. diameter. The two cones have a common vertex and their generatrices are of equal length, so that the circle which is the base of the one cone rolls round that of the other in a plane inclined to it at a constant angle. Draw the plan of the curve traced by a point in the circumference of the rolling cone.*

It is obvious from the mode of generation, that every point in the curve is upon the surface of a sphere having the common vertex for its centre and a generatrix of one of the cones for radius.

When a circle rolls over the circumference of another, so that the two always have a tangent in common at the point of contact, the curve generated by a point in the circumference of the rolling circle is either an epicycloid or a hypocycloid, according as it rolls along the convex or concave side of the fixed circumference.

When the two circles are coplanar the curve generated must of course be in the same plane with them. The curve generated in the above problem is, however, not plane but of double curvature—the *Spherical Epicycloid.*

To construct the plan of the curve:—

(1) Measure a segment of the circumference of the larger fixed circular base equal in length to the whole circumference of the rolling base.

NOTE. The circumferences of circles being directly proportional to their diameters, the required segment will, in this case, be exactly one third of the whole circumference.

(2) Bisect the segment in *o*, and rabat the base of the rolling cone that touches it in that point, about its horizontal trace (*i. e.* about the tangent to the segment at *o*) into H.P., by aid of an auxiliary vertical plane of elevation passing through *o* and the plan, *v*, of the common vertex.

(3) Produce the line *vo* through the centre of the rabatted base to meet its circumference at *P*, and let *P* be the rabatted point that traces the epicycloid. Find its plan,

p, when the base is lifted back to its proper position. This will be one point in the required curve.

(4) Divide half the rabatted base from o to P into a number of equal parts 1, 2, 3, &c., and the half of the segment of the fixed base on the same side into the same number of equal parts 1′, 2′, 3′, &c.

If, now, the rabatted circle roll so that 1 coincides with 1′ the position of P will be one division nearer, and by aid of a new auxiliary vertical plane of elevation through $v1'$ the base may be rotated about the tangent to the segment at the point 1′, and a second point p' in the plan of the epicycloid found.

Similarly, as the rabatted base rolls to the points 2′, 3′, &c., other points in this half of the curve will be determined. The other half may be found by a similar operation on that part of the segment.

NOTE. The student will readily see from his drawing that the vertical plane of elevation first drawn can be made to serve for all the points (1′, 2′, 3′, &c.), and thus somewhat lessen the labour of construction.

23. *To determine a tangent to the spherical epicycloid at a point in the curve.*

The tangent must evidently lie in the tangent-plane to the sphere on which the curve is traced. Also, it can be easily proved that the tangent lies in the tangent-plane to the sphere whose centre is the point of contact of the generating circle, and radius the line drawn from the point of contact to the corresponding position of the tracing point.

Thus, if p' were the plan of the given point, the tangent to the curve would be the common section of a tangent-plane to the first-mentioned sphere at that point, with the tangent-plane at the same point to a sphere whose centre is $1'$ and radius the true length of $1'p'$.

When the spheres are determined, the tangent-line can be drawn through P' at right angles to the plane of the normals to the two spheres at that point. See *General Remarks*, "Tangent-Planes and Normals to Curved Surfaces," Chapter VII.

XIII.

SOLUTIONS OF THE TRIHEDRAL ANGLE.
("SPHERICAL TRIANGLES.")

CASE I.

Given the three faces a, b, *and* c; *to determine the three dihedral angles* A, B, *and* C.

NOTE. In 'spherical triangles' the dihedral angles A, B, and C are those which are opposite to the faces a, b, and c, each to each; namely A to a, B to b, and C to c.

Let the three faces be developed on the horizontal plane of one of them, viz., b.

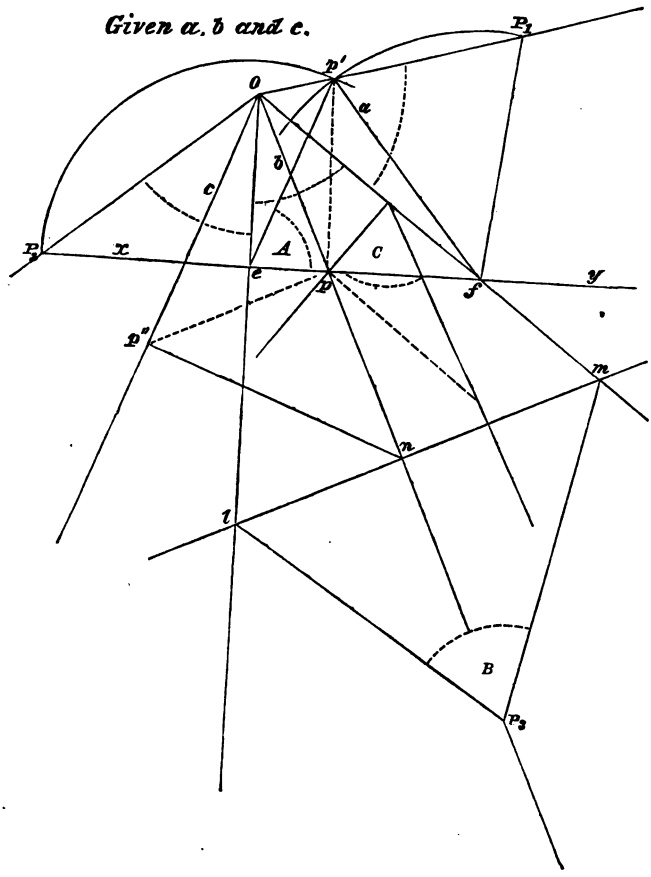
Take a plane of elevation, xy, at right angles to Ol, the common edge of the faces c and b; and let it meet the second edge of face c in point P_2, and cut Ol in e and Om in f.

Make OP_1 on the second edge of a equal to OP_2; since OP_1 and OP_2 are really the same line, viz., the common edge of faces c and a.

Then eP_2, ef, fP_1 are the developed sections of the three faces by the vertical plane through xy.

Rabat the section $ep'f$ in this plane; taking the centre e and radius eP_2, and centre f and radius fP_1, and describing arcs intersecting in p': join $p'e$ and $p'f$. Then p, the plan of P, is determined in xy from p', and Op is the plan of the edge OP on the plane of b. Angle $p'ef$ is the angle A. The plane through p at right angles to Om gives the angle C. And the third angle B is determined by a plane through P, in space, at right angles to the third edge OP, in space, by Problem 15, Chap. III. lm is the horizontal trace of this plane, which is that of the profile angle of B.

Case I.



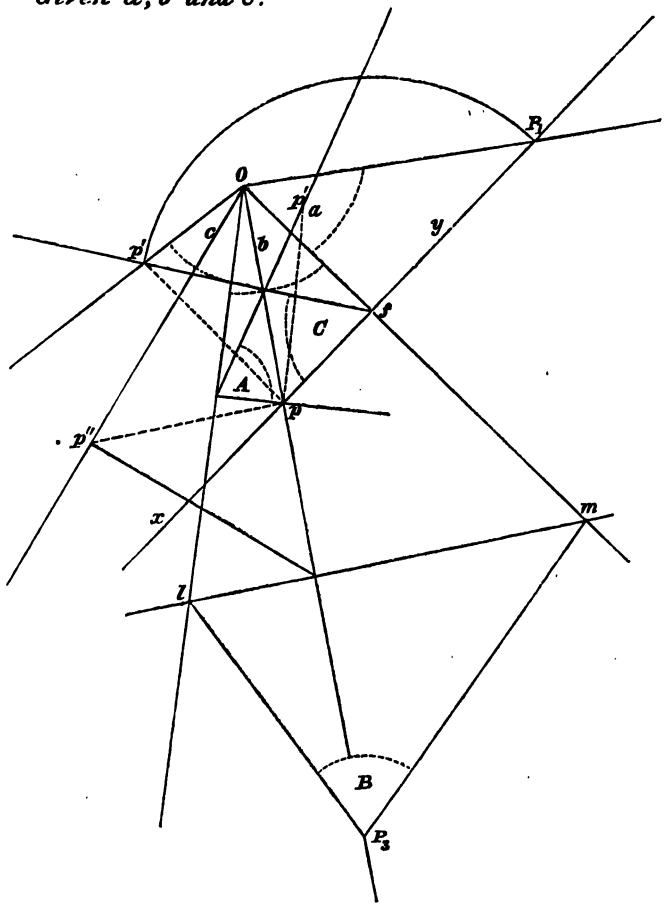
CASE II.

Given the faces a, b, *and the angle* C: *to determine* A, B, *and* c—*i. e. two faces and the included dihedral angle.*

Let the face *a* be developed on the horizontal plane of *b*: since the angle *C* between these faces is known, a plane, *xy*, at right angles to their intersection *Om*, exhibits this angle pfp'; fp' being the vertical trace of the face *a*, the length fp' being made equal to fP_1, p in *xy*, determined from p', gives *Op*, which produced is the indefinite plan of the third edge *OP* on the plane of *b*. A plane through *p* at right angles to the edge *Ol* gives the angle *A*, and enables us to rabat the third face *c* about *Ol*—its horizontal trace. A plane through *P* in space, and at right angles to the edge *OP* in space, and having *lm* for its horizontal trace, will determine the profile angle of *B*.

Case II.

Given a, b and C.



Case III.

Given the dihedral angles B *and* C *and the face* a, *that is, two angles, and their adjacent face: to determine the angle* A *and the faces* b *and* c.

On the horizontal plane take the face *a*, of which *Ol*, *Om* are the edges.

Take the plane *xy* at right angles to *Ol*.

The vertical trace of the face *c* on this plane forms the angle, *B*, with *xy*, which angle being known this trace can be drawn.

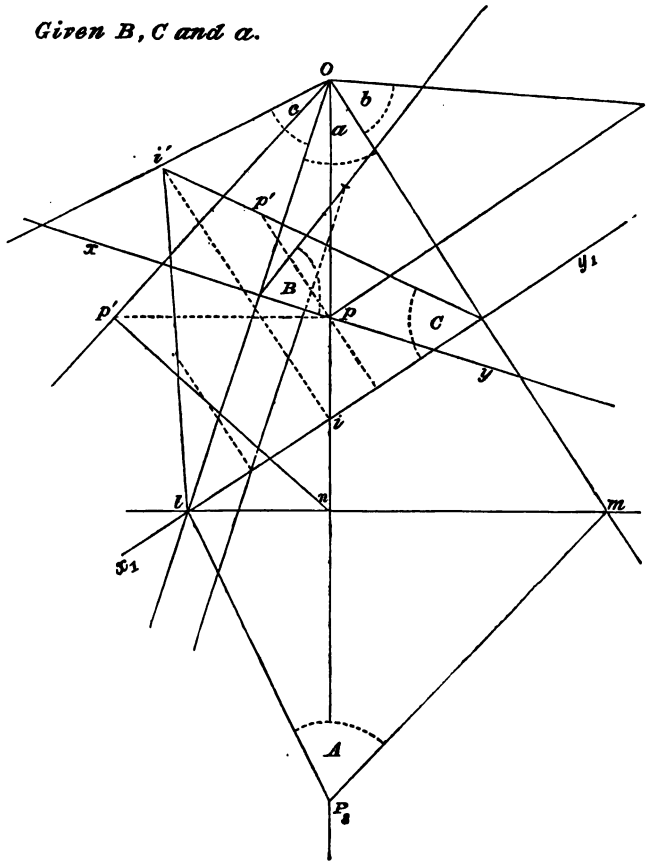
On a plane x_1y_1, at right angles to *Om*, the vertical trace of face *b* forming the known angle, *C*, with x_1y_1 can be drawn.

By the aid of the elevation on *xy* determine the vertical trace of face *c* on plane x_1y_1; the two vertical traces on the latter plane intersect in a point *i'*, whence *i* is found in x_1y_1 and *ii'* is a point in the third edge *OI*, of which draw the indefinite *plan OI*.

The faces *b* and *c* can now be readily rabatted into the plane of *a* by rotation on *Om* and *Ol* respectively. Angle *A* is found by Problem 15, Chapter III., as before, by a plane through point *P* or *I* and perpendicular to *OI*.

CASE III.

Given B, C and a.



CASE IV.

Given two faces a *and* b, *and the dihedral angle,* A, (*that is, two faces, and an angle opposite to one of them*): *to determine the dihedral angles* B *and* C, *and face* c.

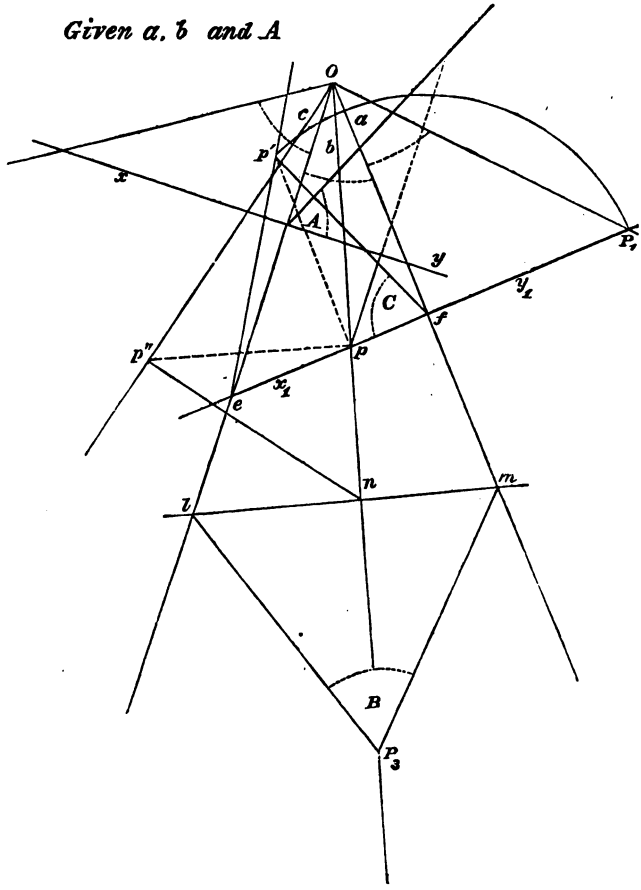
Assume the given face *b* on the horizontal plane, edges *Ol* and *Om*, and face *a* rabatted into that plane about *Om*.

On plane *xy*, at right angles to *Ol*, make the given angle *A* with *xy*, the second line of this angle is the vertical trace of face *c* on this plane.

On a second plane, x_1y_1, assumed at right angles to *Om*, determine the vertical trace ep' of the face *c* by the aid of the former elevation: cut this trace in p' by the circular arc described on plane x_1y_1 by point P_1 on the third edge: join $p'f$, this is the vertical trace of face *a*; obtain p in x_1y_1, then pp' is a second point determined in the third edge *OP* of the spherical triangle. Draw the indefinite plan *Op* and obtain the third angle, *B*, as before, by Problem 15, Chapter III.

CASE IV.

Given a, b and A



CASE V.

Given the dihedral angles B *and* C, *and a face* b *opposite to one of them: to determine faces* a *and* c, *and the third dihedral angle* A.

Let the face *b* be drawn on the horizontal plane, and revolved about its edge *Om* until it makes the angle *C* with that plane.

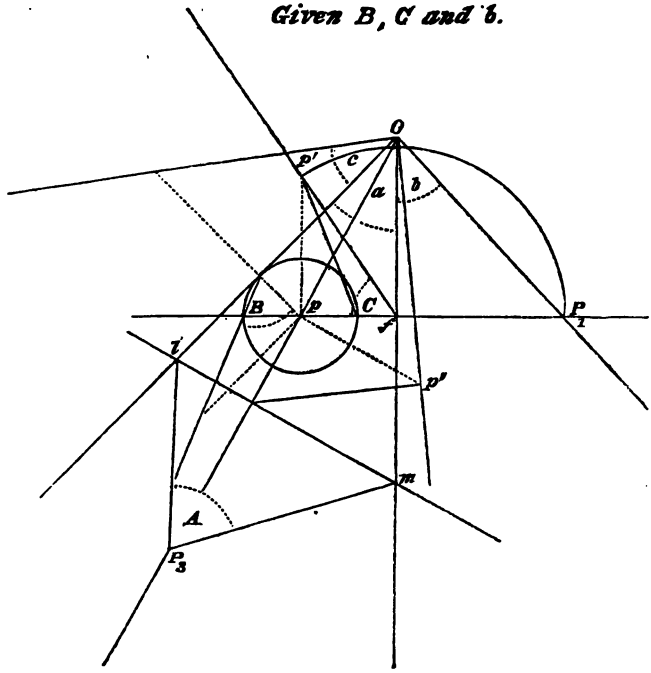
The indefinite line fp' will then be the vertical trace of face *b* on the plane *xy* of the profile angle of *C*; which is, of course, perpendicular to *Om*: and fp' on this trace, taken equal to fP_1, gives the elevation, p', of a point *P* in the second edge of face *b*. p in *xy*, obtained from p', determines the point pp' in this second edge: and Op is the *plan* of that edge.

Determine a plane to contain the line *OP*, in space, just found, and inclined at the known angle of *B*, by Problem 9, Chapter III.: *Ol*, the horizontal trace of this plane, is the third edge of the required spherical triangle.

The face *c* can now be rabatted on the plane of *a* about *Ol*. The angle *A* may be determined as before, by a plane at right angles to *OP* through point *P*.

CASE V.

Given B, C and b.



CASE VI.

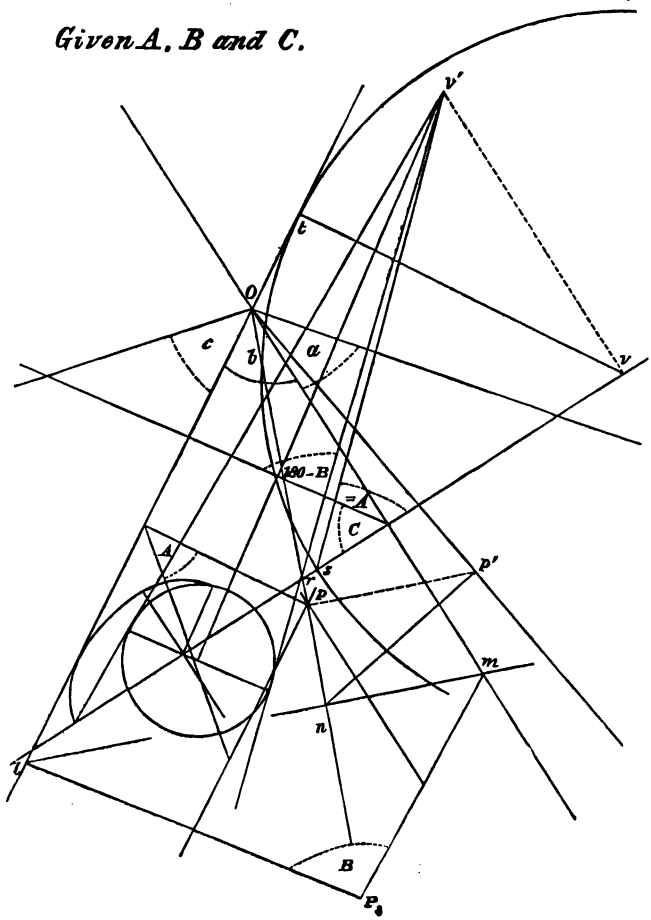
Given the three dihedral angles A, B, *and* C*: to determine the three faces* a, b, *and* c.

If the dihedral angle C has one face in the horizontal plane, the traces of a second face can be readily drawn; and of these the horizontal trace is an indefinite edge of the required spherical triangle.

A third plane making the angle of A with its horizontal, and of $180-B$ with the inclined plane, determined by Problem 26, Chap. III., will complete the required spherical triangle. The intersections Ol, OP, of this third plane with the planes of the angle C, form the second and third edges of the spherical triangle. The face b will thus be determined in the horizontal plane; into which the remaining faces a and c can be rabatted about their respective edges or horizontal traces into the plane of the face b.

CASE VI.

Given A, B and C.



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